

AD-A190 875

UNSTABLE BEHAVIOR OF LASERS AND OTHER OPTICAL SYSTEMS

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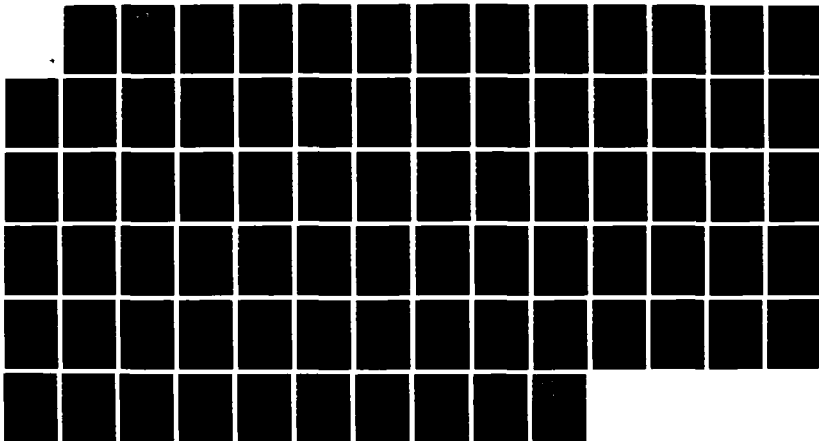
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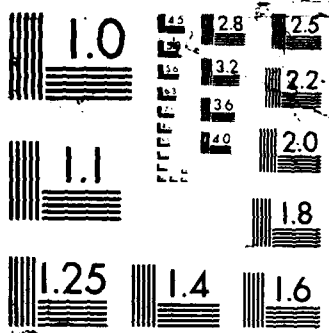
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SECURITY

## DOCUMENTATION PAGE

1a. REF

AD-A190 875

1b. RESTRICTIVE MARKINGS

2a. SEC

1988

3. DISTRIBUTION/AVAILABILITY OF REPORT

2b. DECLASSIFICATION/DOWNGRADING SCHEDULE

Approved for public release;  
distribution unlimited.

4. PERFORMING ORGANIZATION REPORT NUMBER(S)

5. MONITORING ORGANIZATION REPORT NUMBER(S)

ARO 22919.1-PH

6a. NAME OF PERFORMING ORGANIZATION

6b. OFFICE SYMBOL  
(if applicable)

7a. NAME OF MONITORING ORGANIZATION

Drexel University

U. S. Army Research Office

6c. ADDRESS (City, State, and ZIP Code)

7b. ADDRESS (City, State, and ZIP Code)

32 and Chestnut St, Philadelphia, Pa 19104

P. O. Box 12211

Research Triangle Park, NC 27709-2211

8a. NAME OF FUNDING/SPONSORING  
ORGANIZATION8b. OFFICE SYMBOL  
(if applicable)

9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER

U. S. Army Research Office

DAA6-29-85-K-0256

8c. ADDRESS (City, State, and ZIP Code)

10. SOURCE OF FUNDING NUMBERS

P. O. Box 12211

PROGRAM  
ELEMENT NO.PROJECT  
NO.TASK  
NO.WORK UNIT  
ACCESSION NO.

Research Triangle Park, NC 27709-2211

11. TITLE (Include Security Classification)

Unstable behavior of lasers and other optical systems

12. PERSONAL AUTHOR(S)

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13a. TYPE OF REPORT  
final13b. TIME COVERED  
FROM Oct. 85 TO Aug. 8714. DATE OF REPORT (Year, Month, Day)  
November 27, 1987

15. PAGE COUNT 61

16. SUPPLEMENTARY NOTES

The view, opinions and/or findings contained in this report are those of the author(s) and should not be construed as an official Department of the Army position, policy, or decision, unless so designated by other documentation.

17. COSATI CODES

18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)

FIELD

GROUP

SUB-GROUP

Laser Instabilities

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

This final report summarizes the theoretical and experimental activities carried out jointly by the Quantum Optics Group at Drexel University and the Nonlinear Dynamics and Laser Physics Group at Bryn Mawr College during the period October 15, 1985 - August 31, 1987. The report includes detailed reviews of our major findings in the study of laser dynamics, especially theoretically, but also experimentally, and summaries of other parallel studies on Stimulated Raman Scattering (SRS) and non-linear dynamics. Our emphasis has been on the formulation of a generalized version of the Maxwell-Bloch equations for the laser to include transverse degrees of freedom. We have succeeded in developing this model and have investigated in detail its steady state and linear stability properties. Our main result is the discovery of low threshold unstable behaviors and the identification of their physical origin. This

20. DISTRIBUTION/AVAILABILITY OF ABSTRACT

21. ABSTRACT SECURITY CLASSIFICATION

☐ UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT. ☐ DTIC USERS

Unclassified

22a. NAME OF RESPONSIBLE INDIVIDUAL

22b. TELEPHONE (Include Area Code)

22c. OFFICE SYMBOL

# Unstable behavior of Lasers and other Optical Systems

Final report

November 30, 1987

U.S. Army Research Office  
Contract Number DAAG29-85-K-0256

Physics Department  
Drexel University  
Philadelphia, Pa. 19104



Approved for public release;  
Distribution unlimited

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Unannounced	<input type="checkbox"/>
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Date	Spec. Ind.
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## Overview

This final report summarizes the theoretical and experimental activities carried out in joint collaboration by the Quantum Optics Group at Drexel University and the Nonlinear Dynamics and Laser Physics Group at Bryn Mawr College during the period October 15, 1985 - August 31, 1987. Although much of the research conducted under this contract has involved a close interaction between theorists and experimentalists, for convenience we list the main accomplishments under separate headings of theory, experiments and special activities.

The following summaries represent highlights of technical results which are discussed in greater depth in the enclosed publications by the members of our groups.

## The Role of Transverse Effects in Laser Instabilities

### 1. Introduction

One of the unexpected features of early laser systems was the appearance of output pulsations even under steady or nearly steady pumping conditions. In fact, spiking had already been observed in maser experiments<sup>1</sup> even before the discovery of the laser; with the proliferation of solid state optical devices, this effect acquired the status of a nearly universal feature<sup>2</sup>. The interpretation of these phenomena as direct manifestations of intrinsic dynamic instabilities, linked to the nonlinear character of the emission should be ranked among the most significant theoretical advances of the late 1950's and early 1960's<sup>3,4</sup>.

The most common modern framework for the description of laser dynamics is the plane-wave Maxwell-Bloch model, a formulation flexible enough to conform to most experimental conditions and capable of yielding a detailed insight into the laser working mechanism. Yet, for all the important contributions that this model has made to laser physics, it has also shown a consistent pattern of quantitative and even qualitative disagreement with experimental facts, especially when the instabilities become a dominant dynamical feature<sup>5</sup>.

An important starting point for our discussion is the recognition that the plane-wave Maxwell-Bloch equations do predict the existence of instabilities leading to both regular and chaotic pulsations; this is a well established theoretical fact<sup>6</sup>. Thus, homogeneously broadened systems are known to produce output oscillations when operating both in single and multimode configurations, and in and out of resonance, relative to the center of the atomic gain line. Of special significance is the existence of an isomorphism between the single-mode laser model and the Lorenz equations<sup>6a</sup>. The Lorenz equations, originally derived to simulate the onset of convective instabilities<sup>7</sup>, are well known in the mathematical literature as the paradigm of deterministic chaos<sup>8</sup>. Thus, the isomorphism between the laser and the Lorenz equations supports the notion that the laser itself can be the source of chaotic behavior.

While the link between the observed laser pulsations and dynamic instabilities is a very reasonable theoretical proposition, the prediction that these phenomena can occur only for very large values of the pump parameter, especially under resonant conditions, is the source of major difficulties with the interpretation of experimental results. Indeed, if one took the theoretical threshold values too seriously, one would have to conclude that laser instabilities are observable only under exceptional circumstances.

An additional difficulty with the plane-wave Maxwell-Bloch equations is the unconditional stability of their rate equations limit. Accepting this result leads to the conclusion that all laser instabilities are manifestations of atomic coherence, this assertion is hardly in agreement with the behavior of Ruby, Nd:YAG, CO<sub>2</sub> and other lasers for which the validity of the rate equations description is well justified.

Better agreement between theory and experiments is obtained when the laser undergoes a phase instability<sup>6g</sup>, a phenomenon that requires operation under detuned conditions and which is responsible, for example, for the observed mode hopping in CO<sub>2</sub> laser systems<sup>9</sup>. However, the physical mechanism that drives a phase instability is quite different from that which is operative under resonant conditions (i.e. the amplitude instability)<sup>10</sup> so that the theory of phase instabilities is not very helpful in clarifying the nature of the "high second threshold problem". The theory of inhomogeneously broadened lasers is also in good qualitative agreement with the observed unstable phenomena<sup>11</sup>; in this case, the lowering of the second laser threshold to the observed experimental range<sup>12</sup> is probably due to the large increase in the number of degrees of freedom that are needed to describe the laser dynamics, and offers little clue as to the failing of the homogeneously broadened model.

It is probably fair to say that the plane-wave approximation, long viewed as adequate in capturing the essential aspects of laser dynamics, may be a much stronger theoretical constraint than anticipated. In fact, important warning signals in this direction came from the lack of quantitative agreement between the predictions of the plane-wave studies of optical bistability<sup>13</sup> and

the careful absolute measurements performed by Kimble and collaborators under steady state conditions<sup>14</sup>. If, however, these theoretical propositions were in disagreement with the experiments in a quantitative way, more serious qualitative discrepancies have emerged from the study of self-pulsing in bistable systems. Here the plane-wave theory predicts a rich variety of dynamical behaviors<sup>15</sup> which do not appear to have a counterpart among the observed patterns<sup>16</sup>.

The main objective of this research is to produce evidence that transverse degrees of freedom play a far more important role in laser dynamics than might be expected. In fact, we have known for some time that transverse effects can have a strong influence on the stationary and dynamic behavior of passive driven systems<sup>17</sup> and that plane-wave and transverse models can approach chaos by way of very different routes<sup>18</sup>. The combined message of a small but growing number of experimental<sup>19</sup> and theoretical reports<sup>20</sup> supports the belief that effects related to a departure from the plane-wave configuration may be even more influential when the optical resonator contains an active rather than a passive medium. If we accept this premise, the inclusion of transverse degrees of freedom into the existing theories becomes a necessary requirement for a successful description of realistic laser systems.

Indeed, several theoretical studies have already attempted to improve on the plane-wave description. An analysis of the modes and of their spatial stability in a realistic resonator can be found, for example, in Ref. 21. Reference 22 contains a study of the emission frequency of an inhomogeneously broadened laser when the field configuration is no longer of the pure Gaussian type. The problem of laser dynamics in the presence of transverse effects was addressed in Ref. 23 using an approximate set of equations and evidence was presented for the appearance of self-pulsing.

A useful alternative approach is based on the assumption that the cavity field maintains a fixed Gaussian profile during the evolution<sup>17b,24</sup>. On the surface this appears to be a rather crude approximation because the presence of an active medium may be expected to introduce significant distortion in the transverse pattern of the field. In the case of passive systems, however, this



approach has yielded quite reasonable results<sup>25</sup>, perhaps because of the constraints that the injected mode-matched signal imposes on the cavity field. The situation is likely to be more critical with active systems, so that a more realistic approach may be required. In fact, even the simple Gaussian model of the laser yields remarkable deviations from the plane-wave behavior; the Risken and Nummedal instability<sup>6b,c</sup> of the longitudinal sidebands, for example, is suppressed under resonant conditions if the cavity field has a  $TEM_{00}$  profile<sup>24</sup>.

In this study we consider those physical and geometrical features that are absent by definition from the plane-wave theory. To be specific, we focus on the effects of diffraction, on the wavefront curvature induced by the spherical mirrors, and on the transverse and longitudinal gain variations caused by the pump mechanism. It would also be very desirable to consider the effects of limiting intracavity apertures, but this aspect of the problem is more complicated and has been ignored in our study.

Of course the mathematical description of this problem is considerably more involved than in the case of a plane-wave model, and significant theoretical progress is likely to require large-scale numerical calculations when the parameters of the system take on arbitrary values. An unavoidable drawback of most numerical approaches is that a global understanding usually requires the analysis of a wide region of the parameter space. For this reason it is much more desirable to investigate transverse effects by analytical means, as much as feasible. Of course this cannot be done for arbitrary configurations of the laser system. Fortunately it is possible to identify a non-trivial setting that allows significant theoretical progress with a minimum of numerical labor. Our approach is based on an appropriate extension of the well known uniform field limit<sup>26</sup>, a model that has played a major role in defining our current understanding of the plane-wave theories of active and passive systems. In spite of its idealized nature, the uniform field limit of the plane-wave model has led to predictions that are in remarkably close qualitative agreement with the results of the exact linear stability analysis<sup>27</sup> and with the numerical solutions of the Maxwell-Bloch equations<sup>28</sup>. It is hoped that the same conclusion will remain valid in the presence

of transverse effects. Much more work, however, will be needed to confirm this conjecture.

An important consequence of this limit, as we show in this paper, is that the steady state solutions are of the single-mode type, a conclusion which also implies that the empty cavity modes are also exact modes of the filled cavity. This statement is true in the plane-wave approximation, and remains valid in the presence of transverse effects. Thus, at once, the mathematical description of the problem is simplified to a large extent, and earlier studies based on the single transverse mode approximation<sup>17e,24</sup> can be understood in more rigorous terms.

With the assumption that the cavity linewidth is much smaller than the atomic decay rates, the so called "good cavity" limit, we can describe the emergence of unstable states through a development that rests almost entirely on analytic grounds; in fact, only the final evaluation of the rates of evolution of the fluctuation variables requires a numerical computation. The linear stability analysis leads to one of the most interesting results: the instability thresholds for a fairly realistic cavity design are much closer to the first laser threshold than predicted by the plane-wave theory, and are in much better quantitative agreement with the experimental picture. For example, under appropriate conditions, unstable operation emerges already when the pump parameter is only a few percent above threshold, while the plane-wave theory requires pump parameter values at least ten times larger than needed to produce laser emission.

We show, in addition, that low threshold instabilities can be found also in the full adiabatic elimination limit, i.e. when the dynamics of the laser is described entirely by the field equation (thus "a fortiori" the same conclusion holds in the rate equation limit when only the polarization is eliminated adiabatically). According to this result instabilities are not a necessary consequence of atomic coherence, in contrast with the predictions of the plane-wave theory.

The physical setting of the linear stability analysis is the following: with the laser operating in steady state, we look for the existence of unstable modes, as evidenced by the appearance of an exponential growth for the field fluctuations. This approach to the problem is conceptually identical to the one adopted by Risken and Nummedal<sup>6b,c</sup> in their classic study of multimode instabilities.

although there are also significant differences, as we shall discuss in the main body of this report.

In Section 2 we construct the empty cavity eigenfunctions and eigenfrequencies for a folded ring resonator. In Section 3 we describe the extended Maxwell-Bloch model, we introduce the uniform field limit, and calculate the steady state configuration of the system which we then analyze for different values of the parameters. In section 4 we outline the details of the linear stability analysis. In Section 5 we discuss the results of this calculation, give evidence for the existence of low threshold instabilities, and show how these effects persist even after adiabatic elimination of all the atomic variables. We conclude, in Section 6, with a brief overview and some general comments. A preliminary account of this work was published in Ref. 29.

## 2. The empty cavity modes

This section contains a brief description of a calculation leading to the modal eigenfunctions and eigenfrequencies of an empty resonator in the paraxial approximation<sup>30</sup>. Its purpose is to provide a self-contained introduction to the basic tools that have been used in our subsequent calculations.

The analysis of an empty resonator of length  $\Lambda$ , as shown in Fig. 1, begins with the free-field wave equation

$$\nabla_2^2 E(r,z,t) - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} E(r,z,t) = 0 \quad (2.1)$$

where  $z$  is the coordinate along the axial direction of propagation of the field and  $r$  is the radial coordinate measured from the axis of symmetry of the system. As implied by Eq. (2.1) we consider a system with cylindrical symmetry.

In the paraxial approximation we seek elementary solutions of the form

$$E \equiv E_p(r,z,t) = A_p(r,z) e^{i(k_p z - \omega_p t)} \quad (2.2)$$

In fact, the solutions corresponding to a cylindrically symmetric system should be labelled by two

indices related to the axial and radial degrees of freedom, respectively. We will omit the axial modal index, for simplicity, except where necessary.

In the slowly varying amplitude approximation, and in terms of the scaled coordinates

$$\eta = z/L, \quad \rho = \sqrt{\frac{\pi}{L\lambda_p}} r \quad (2.3)$$

where  $L$  is the distance between the curved mirrors, the wave equation takes the form

$$\frac{\partial A_p}{\partial \eta} = \frac{i}{4} \left( \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} \right) A_p \quad (2.4)$$

whose solutions are

$$A_p(\rho, \eta) = \frac{2}{v(\eta)} L_p \left( \frac{2\rho^2}{v^2(\eta)} \right) \exp \left( -\frac{\rho^2}{v^2(\eta)} \right) \exp \left\{ i \left[ \frac{\rho^2}{u(\eta)} - (2p+1) \tan^{-1} \eta/\eta_0 \right] \right\} \\ p = 0, 1, 2, \dots \quad (2.5)$$

The origin of the longitudinal axis is selected at point 0 of Fig. 1 where  $E_p(\rho, \eta, t)$  has a plane-wave structure. The functions  $v(\eta)$  and  $u(\eta)$  are defined by

$$v(\eta) = \sqrt{\eta_0} \sqrt{1 + (\eta/\eta_0)^2} \quad (2.6a)$$

$$u(\eta) = \frac{1}{\eta} (\eta^2 + \eta_0^2) \quad (2.6b)$$

and  $\eta_0 = v^2(0) \equiv v_0^2$  is an arbitrary parameter which must be calculated in conformity with the cavity geometry. Note that with the selected scaling (2.3),  $\eta_0$  is the Rayleigh range of the Gauss-Laguerre beam (2.5), while  $\sqrt{\eta_0}$  is the beam waist.

The assignment of  $\eta_0$  stems from the following considerations. The beam parameter  $q(\eta)$  defined by

$$\frac{1}{q(\eta)} = \frac{1}{u(\eta)} + \frac{i}{v^2(\eta)} \quad (2.7)$$

satisfies elementary mapping rules under translations and reflections from the curved surfaces.

Beginning with a given beam parameter  $q_0$  at position 0, we can easily construct the beam

parameter  $q_4$  after one full loop (Fig. 1). The condition  $q_4 = q_0$  secures that the beam parameter after one round trip acquires the same starting value.

For simplicity we select a cavity with a high degree of longitudinal symmetry as shown in Fig. 1. This leads to<sup>30</sup>

$$\eta_0 = v_0^2 = \frac{1}{2} \sqrt{\frac{(\rho_0 - 1)(\rho_0 - 1 + f)}{f(1 + \rho_0) - 1}} \quad (2.8)$$

where  $\rho_0 = R/L$  is the scaled radius of curvature of the spherical mirrors and the parameter  $f$  is defined by  $L/\Lambda$ .

In a similar way we can construct the analytic form of the modal functions  $A_p(\rho, \eta)$  for  $1/2 \leq \eta \leq \Lambda/L - 1/2$ . The beam waist in this case is given by

$$v_2^2 = \frac{1}{\text{Im}\left(\frac{1}{q_2}\right)}$$

where  $q_2$  is the beam parameter at  $\eta = \Lambda/2L$  (point 2 of Fig. 1). The explicit form of the modal functions in this range is

$$A_p(\rho, \eta) = B_p[q_2(\eta - \Lambda/2L)] \exp\left[-i(2p+1) \tan^{-1}\left(\frac{\eta - \Lambda/2L}{v_2^2}\right)\right] e^{i\varphi} \quad (2.9)$$

where, for convenience, we have defined

$$B_p(q_2(\eta)) = \frac{2}{v_2(\eta)} L_p\left(\frac{2\rho^2}{v_2^2(\eta)}\right) \exp\left(-\frac{\rho^2}{v_2^2(\eta)} + i \frac{\rho^2}{u_2(\eta)}\right) \quad (2.10)$$

The functions  $v_2(\eta)$  and  $u_2(\eta)$  are defined by Eqs. (2.6) with  $\eta_0$  replaced by

$$v_2^2 = v^2(\Lambda/2L) = \frac{1}{\rho_0^2 \eta_0} \left[ \eta_0^2 \left( \frac{1}{f} - 1 - \rho_0 \right)^2 + \frac{1}{4} \left( \frac{\rho_0 - 1}{f} + 1 \right)^2 \right] \quad (2.11)$$

The phase factor  $\exp(i\varphi)$  measures the shift introduced by the free propagation along the segment  $1' \rightarrow 2$  in Fig. 1. This phase factor can be calculated by matching Eq. (2.9) at position  $1'$  (i.e.  $\eta = 1/2$ ) with the field given by Eq. (2.5) after a reflection from the first spherical mirror. The

reflection for near normal incidence is simulated by the operator

$$\mathfrak{R} = \exp\left(-i \frac{2\rho^2}{\rho_0}\right) \quad (2.12)$$

It follows at once that

$$\varphi = -(2p+1) \left[ \tan^{-1}\left(\frac{1}{2v_0^2}\right) + \tan^{-1}\left(\frac{1/f-1}{2v_2^2}\right) \right] \quad (2.13)$$

In addition, after a full loop, the empty cavity solution  $A_p$  at position 4 is related to the solution at position 0 by the equation

$$A_p(4) = e^{2i\varphi} A_p(0) \quad (2.14)$$

Now we calculate the empty cavity resonances. The boundary condition for the elementary radial solution of order  $p$  is

$$E_p(\rho, 0, \tau) = E_p(\rho, \Lambda/L, \tau) \quad (2.15)$$

or

$$A_p(0) = e^{i\omega_p \Lambda/c} A_p(4) \quad (2.16)$$

when expressed in terms of the modal amplitudes  $A_p$ . From Eqs. (2.16), (2.14) and (2.13) we have

$$\omega_{n,p} \frac{\Lambda}{c} = 2\pi n + 2(2p+1) \left[ \tan^{-1}\left(\frac{1}{2v_0^2}\right) + \tan^{-1}\left(\frac{1/f-1}{2v_2^2}\right) \right] \quad (2.17)$$

with  $n = 0, \pm 1, \pm 2, \dots$ , and  $p = 1, 2, \dots$ . The eigenfrequencies are labelled by two indices; an axial index  $n$  and a transverse index  $p$ . The longitudinal intermode spacing is given by the usual formula  $2\pi nc/\Lambda$  while the frequency separation between adjacent transverse modes is

$$(\omega_{n,p+1} - \omega_{n,p}) \equiv \delta\omega_1 = 4 \frac{c}{\Lambda} \left[ \tan^{-1}\left(\frac{1}{2v_0^2}\right) + \tan^{-1}\left(\frac{1/f-1}{2v_2^2}\right) \right] \quad (2.18)$$

An important property of the modal functions  $A_p(\rho, \eta)$  is that for every value of  $\eta$  they form an orthonormal set in the sense that

$$\int_0^{\infty} d\rho \rho A_p^*(\rho, \eta) A_{p'}(\rho, \eta) = \delta_{p,p'} \quad (2.19)$$

With the additional reasonable assumption that  $\{A_p(\rho, \eta)\}$  form a complete set with respect to the radial coordinate, these functions provide a useful basis for the expansion of any cavity field at every point along the  $\eta$  axis. This property will play an essential role in the following development.

### 3. The generalized Maxwell-Bloch problem: description of the model and the steady state solution

#### 3.1 Derivation of the equations

We extend the traditional description of the laser with the inclusion of several features that are omitted, by definition, from the plane-wave theory:

- i) the diffraction caused by the finite transverse cross section of the field and by its radial variations of amplitude and phase;
- ii) the wavefront curvature imposed by the presence of reflecting spherical surfaces;
- iii) the longitudinal and radial variations of the equilibrium population difference.

We do so in the context of the paraxial approximation and under the assumption of cylindrical symmetry for the field and for the atomic variables. The latter restriction, in particular, is a compromise dictated by the need for including transverse degrees of freedom with the lowest level of analytic and numerical complexity. We recognize that this is not a very realistic setting for a typical laser but, nevertheless, we expect that it will provide useful insights into the role of transverse effects.

One can model the effects of diffraction by adding a term of the form  $\nabla_{\perp} F$  to the usual wave equation for the slowly varying amplitude  $F$  where

$$\nabla_{\perp} = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \quad (3.1)$$

is the transverse Laplacian for an axially symmetric geometry. The curved reflectors affect the boundary conditions indirectly by way of the dependence of the radial eigenfrequencies on the radius of curvature of the mirrors (see Eq. (2.17)) as we shall discuss more explicitly below. Finally, the longitudinal and transverse variations of the pump are reflected in the equilibrium value of the population difference which is no longer uniform along the transverse direction as in the plane-wave model.

With these modifications the Maxwell-Bloch equations take the form

$$\frac{\partial F}{\partial \eta} + \frac{1}{v} \frac{\partial F}{\partial \tau} = \frac{i}{4} \left( \frac{\partial^2 F}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial F}{\partial \rho} \right) - \alpha L P \quad (3.2a)$$

$$\frac{\partial P}{\partial \tau} = -(FD + (1+i\bar{\delta}_{AC})P) \quad (3.2b)$$

$$\frac{\partial D}{\partial \tau} = -\bar{\gamma} \left\{ -\frac{1}{2} (F^* P + F P^*) + D - \chi(\rho, \eta) \right\} \quad (3.2c)$$

We select as the cavity reference frequency  $\omega_0$  the  $p=0$  empty cavity resonance,  $\omega_{n,0}$ , so that the slowly varying field amplitude  $F$  is related to the Maxwell field by

$$E(r, z, t) = \frac{\hbar \sqrt{\gamma_{\perp} \gamma_{\parallel}}}{2\mu} \frac{1}{2} \left( F(r, z, t) e^{i(k_0 z - \omega_0 t)} + \text{c.c.} \right) \quad (3.3)$$

where  $\mu$  is the modulus of the atomic transition dipole moment and  $\gamma_{\perp}$  and  $\gamma_{\parallel}$  are the polarization and population decay rates, respectively, and  $k_0 = \omega_0/c$ . The spatial coordinates  $\eta$  and  $\rho$  are defined by Eqs. (2.3) with  $\lambda_p$  replaced by  $\lambda_0$  (the reference wavelength);  $\tau$ ,  $v$  and  $\bar{\delta}_{AC}$  are defined by



$$\tau = \gamma_{\perp} t, \quad v = \frac{c}{L\gamma_{\perp}}, \quad \bar{\delta}_{AC} = \frac{\omega_A - \omega_0}{\gamma_{\perp}} \quad (3.4)$$

where  $\omega_A$  is the atomic transition frequency; the parameter  $\bar{\gamma}$  denotes the ratio  $\gamma_{\parallel}/\gamma_{\perp}$  and the function  $\chi(\rho, \eta)$  simulates the longitudinal and transverse variations of the pump. There is a good deal of flexibility with the choice of  $\chi$  depending on the details of the excitation system. For definiteness we select

$$\chi(\rho, \eta) = \begin{cases} \bar{\chi}(\rho) & |\eta| < \frac{L_A}{2L} \\ 0 & \text{otherwise} \end{cases} \quad (3.5a)$$

where  $L_A$  denotes the length of the medium, and consider, as two possible models, the radial functions

$$\bar{\chi}(\rho) = \exp\left(-\frac{\rho^2}{2\rho_p^2}\right) \quad (3.5b)$$

and

$$\bar{\chi}(\rho) = 2 \exp\left(-\frac{\rho^2}{2\rho_p^2}\right) - 1 \quad (3.5c)$$

In the first case the active medium is transparent at the outer boundaries of the pumped volume (this model is plausible, for example, in the case of a four level system such as Nd:YAG), while in the second a lowering the pump strength creates an absorbing region, such as one may find in a Ruby laser, for example.

A way to state the boundary conditions is to require that the Maxwell field at position  $\eta=0$  matches the field that has reached the position  $\eta=L$  after propagating one full loop through the cavity. Formally this implies

$$E(\rho, 0, \tau) = E(\rho, L, \tau) \quad (3.6a)$$

or, in terms of the slowly varying amplitude

$$F(\rho, 0, \tau) = F(\rho, \Lambda/L, \tau) e^{i \omega_0 \Lambda/c} \quad (3.6b)$$

Equation (3.6b) must be understood as the assignment that sets the slowly varying field amplitude at the boundary  $\eta=0$  and at time  $\tau$  in terms of the amplitude that has just completed one loop through the cavity according to Eq. (3.2), and in compliance with the appropriate reflections at the curved surfaces. This type of boundary condition is especially appropriate in connection with the numerical solution of Eqs. (3.2). In the following we carry out our analysis of the steady state and of the linear stability in terms of a modal expansion of the field so that a different version of the boundary conditions is required.

As our starting point we let

$$F(\rho, \eta, \tau) = e^{-i \bar{\delta}\Omega \tau} \bar{F}(\rho, \eta, \tau) = e^{-i \bar{\delta}\Omega \tau} \sum_p A_p(\rho, \eta) f_p(\eta, \tau) \quad (3.7a)$$

$$P(\rho, \eta, \tau) = e^{-i \bar{\delta}\Omega \tau} \bar{P}(\rho, \eta, \tau) \quad (3.7b)$$

where  $\bar{\delta}\Omega$  is the offset between the unknown operating laser frequency and the reference frequency  $\omega_0$  in units of  $\gamma_\perp$ . We substitute Eqs. (3.7) into the Maxwell-Bloch equations (3.2) and integrate over the radial variable with the help of the orthogonality integral (2.19). The result is the following set of coupled equations for the modal amplitudes  $f_p(\eta, \tau)$  and for the atomic variables<sup>17e</sup>

$$\frac{\partial f_p}{\partial \eta} + \frac{1}{v} \frac{\partial f_p}{\partial \tau} = i \frac{\bar{\delta}\Omega}{v} f_p - \alpha L \int_0^\infty d\rho \rho A_p^*(\rho, \eta) \bar{P}(\rho, \eta, \tau) \quad (3.8a)$$

$$\frac{\partial \bar{P}}{\partial \tau} = -(\bar{F}D + (1+i\bar{\Delta})\bar{P}) \quad (3.8b)$$

$$\frac{\partial D}{\partial \tau} = -\bar{\gamma} \left\{ -\frac{1}{2} (\bar{F}^* \bar{P} + \bar{F} \bar{P}^*) + D - \chi(\rho, \eta) \right\} \quad (3.8c)$$

where  $\bar{\Delta} = \bar{\delta}_{AC} - \bar{\delta}\Omega$ . The modal amplitudes obey the boundary conditions

$$f_p(-1/2, \tau) = R e^{-i \delta_p} e^{i \bar{\delta}\Omega \gamma_\perp \frac{\Lambda-L}{c}} f_p(1/2, \tau - \gamma_\perp \frac{\Lambda-L}{c}) \quad (3.8d)$$

where  $\delta_p \equiv (\omega_{n,p} - \omega_0)\Lambda/c$  and  $R$  denotes the reflectivity of the mirrors; a brief outline of the derivation of Eq. (3.8d) is given in Appendix A. Note that the sum in Eq. (3.7a) extends to all nonnegative integers  $p$ . In practice, however, transverse modes with  $p$  greater than 1 or 2 suffer significant diffraction losses typically because of intracavity apertures. We do not have provisions in our treatment for describing this type of losses; we must keep in mind, however, that only very few transverse modes are expected to play a significant dynamical role in the evolution of a real laser.

In steady state the atomic variables are given by

$$\bar{P}_{st} = -\chi \bar{F}_{st} \frac{1 - i\bar{\Delta}}{1 + \bar{\Delta}^2 + |\bar{F}_{st}|^2} \quad (3.9a)$$

$$D_{st} = \chi \frac{1 + \bar{\Delta}^2}{1 + \bar{\Delta}^2 + |\bar{F}_{st}|^2} \quad (3.9b)$$

and the equation for the modal amplitudes becomes

$$\frac{\partial f_p}{\partial \eta} = i \frac{\bar{\delta}_p \Omega}{v} f_p + \alpha L (1 - i\bar{\Delta}) \int_0^\infty d\rho \rho A_p^*(\rho, \eta) \chi(\rho, \eta) \frac{\bar{F}_{st}}{1 + \bar{\Delta}^2 + |\bar{F}_{st}|^2} \quad (3.10)$$

where

$$\bar{F}_{st}(\rho, \eta) = \sum_p A_p(\rho, \eta) f_p(\eta) \quad (3.11)$$

Even the steady state problem, in general, offers formidable analytic difficulties. Important simplifications follow in the limit

$$\alpha L_A \rightarrow 0, \quad T \rightarrow 0; \quad 2C = \frac{\alpha L_A}{T} = \text{arbitrary} \quad (3.12a)$$

and

$$|\bar{\delta}_p - 2\pi m| = O(1) \quad (3.12b)$$

for all  $p \neq 0$  and  $m=0, \pm 1, \pm 2, \dots$ . The symbol  $T = 1 - R$  denotes the mirror transmittivity. Equation

(3.12a) is well known from earlier studies of optical instabilities in the plane wave regime<sup>26</sup>. Equation (3.12b) implies that the frequency separation between adjacent radial modes should be of the same order of magnitude as the frequency spacing between longitudinal resonances, and simultaneously excludes degeneracies or quasi-degeneracies between transverse and longitudinal modes. The two conditions (3.12) taken together define the extended uniform field limit for the chosen laser system. Of course, it is understood that the label "uniform field limit" should not be taken literally. What is actually independent of  $\eta$ , in this limit, is the set of modal amplitudes  $\{f_p\}$ . The field profile inside the cavity is assigned by the modal functions  $\{A_p\}$  which, of course, depend on the longitudinal coordinate  $\eta$ .

The importance of the uniform field limit in this context rests on its ability to provide a strong analytic contribution to the solution of the laser equations of motion and a deeper view into the role that the many parameters play in setting the stationary and dynamical properties of the system.

As we shall prove below, the stationary state of the laser field in the uniform field limit is of the single longitudinal and transverse mode type; as a consequence, it follows that the empty cavity eigenfunctions, in this limit, are also modes of the resonator in the presence of the active medium.

We now return our attention to the solution of the stationary modal equations (3.10) under the conditions (3.12). The spatial integral of Eq. (3.10) yields

$$f_p(1/2) - f_p(-1/2) = i \frac{\bar{\delta}\Omega}{v} \int_{-1/2}^{+1/2} d\eta f_p + \alpha L (1 - i\bar{\Delta}) \int_{-1/2}^{+1/2} d\eta \int_0^\infty d\rho \rho A_p^* \chi \frac{\bar{F}_{st}}{1 + \bar{\Delta}^2 + |\bar{F}_{st}|^2} \quad (3.13)$$

Assuming that  $\delta\Omega$  is of the order of  $T$ , the boundary conditions (3.8) in steady state can also be cast into the approximate forms

$$(p=0); \quad f_0(1/2) - f_0(-1/2) = T \left( 1 - i \frac{\bar{\delta}\Omega}{T} \gamma_\perp \frac{\Lambda - L}{c} \right) f_0(1/2) \quad (3.14a)$$

$$(p \neq 0); \quad f_p(1/2) - f_p(-1/2) = \left( 1 - R e^{-i\delta_p} e^{i\bar{\delta}\Omega \gamma_\perp \frac{\Lambda - L}{c}} \right) f_p(1/2) \quad (3.14b)$$

If we combine Eqs. (3.14) with Eq. (3.12) we obtain

$$(p=0); \quad f_0(1/2) - f_0(-1/2) = T \left( 1 - i \frac{\bar{\delta}\Omega}{T} \gamma_{\perp} \frac{\Lambda-L}{c} \right) f_0(1/2) =$$

$$i \frac{\bar{\delta}\Omega}{v} \int_{-1/2}^{+1/2} d\eta f_0 + \alpha L (1 - i\bar{\Delta}) \int_{-1/2}^{+1/2} d\eta \int_0^{\infty} d\rho \rho A_0^* \chi \frac{\bar{F}_{st}}{1 + \bar{\Delta}^2 + |\bar{F}_{st}|^2} \quad (3.15a)$$

$$(p \neq 0); \quad f_p(1/2) - f_p(-1/2) = \left( 1 - R e^{-i\delta_p} e^{i \frac{\bar{\delta}\Omega}{T} \gamma_{\perp} \frac{\Lambda-L}{c}} \right) f_p(1/2) =$$

$$i \frac{\bar{\delta}\Omega}{v} \int_{-1/2}^{+1/2} d\eta f_p + \alpha L (1 - i\bar{\Delta}) \int_{-1/2}^{+1/2} d\eta \int_0^{\infty} d\rho \rho A_p^* \chi \frac{\bar{F}_{st}}{1 + \bar{\Delta}^2 + |\bar{F}_{st}|^2} \quad (3.15b)$$

Equations (3.15) lead to the following conclusions:

- (i) The difference  $f_0(1/2) - f_0(-1/2)$  is of the order of  $T$ , so that  $f_0$  is uniform along the axial direction;
- (ii) the amplitudes  $f_p$  with  $p \neq 0$  is of the order of  $T$ .

Hence, it follows that in the uniform field limit a possible steady state solution is characterized by the modal amplitudes  $f_0 = O(1)$ ,  $f_{p \neq 0} = O(T)$  and the cavity field configuration is of the  $TEM_{00}$  type.

The immediate consequence of these considerations is the state equation

$$1 - i \frac{\bar{\delta}\Omega}{T} \gamma_{\perp} \frac{\Lambda}{c} = 2C(1 - i\bar{\Delta}) \frac{L}{L_A} \int_{-L_A/2}^{+L_A/2} d\eta \int_0^{\infty} d\rho \rho \bar{\chi} \frac{|A_0(\rho, \eta)|^2}{1 + \bar{\Delta}^2 + |A_0(\rho, \eta) f_0(\eta)|^2} \quad (3.16)$$

for the modal amplitude  $f_0$ , where we have taken Eq. (3.5a) into account. This can be split into real and imaginary parts with the result

$$1 = 2C \frac{L}{L_A} \int_{-\frac{L_A}{2L}}^{+\frac{L_A}{2L}} d\eta \int_0^\infty d\rho \rho \bar{\chi} \frac{|A_0(\rho, \eta)|^2}{1 + \bar{\Delta}^2 + |A_0(\rho, \eta) f_0(\eta)|^2} \quad (3.17a)$$

$$\bar{\delta}\Omega = \bar{\kappa} \bar{\Delta} \quad \text{or} \quad \bar{\delta}\Omega = \frac{\bar{\kappa}}{1 + \bar{\kappa}} \bar{\delta}_{AC} \quad (3.17b)$$

where  $\bar{\kappa} = cT/\Lambda\gamma_\perp$  denotes the cavity linewidth in units of  $\gamma_\perp$ . Equation (3.17b) is an alternative form of the usual mode-pulling formula which remains unchanged in the uniform field limit even in the presence of transverse effects.

The stationary state (3.17) is not unique because, depending on the gain parameter  $2C$ , other steady state configurations are also possible. In fact with a simple extension of the above argument, as shown in Appendix B, one can derive the following more general result for a given fixed value of  $\bar{p}$

$$1 = 2C \frac{L}{L_A} \int_{-\frac{L_A}{2L}}^{+\frac{L_A}{2L}} d\eta \int_0^\infty d\rho \rho \bar{\chi} \frac{|A_{\bar{p}}|^2}{1 + \bar{\Delta}^2 + |A_{\bar{p}} f_{\bar{p}}|^2} \quad (3.18a)$$

$$\bar{\delta}\Omega_{\bar{p}} = \frac{\bar{\kappa}}{1 + \bar{\kappa}} \bar{\delta}_{AC} + \frac{\bar{\alpha}_1}{2\pi} \frac{\bar{\delta}_{\bar{p}}}{1 + \bar{\kappa}} \quad (3.18b)$$

where  $\bar{\alpha}_1$  is the longitudinal mode spacing in units of  $\gamma_\perp$  ( $\bar{\alpha}_1 = 2\pi c/\Lambda$ ). The steady state configuration corresponding to Eqs. (3.18) is also of the single-mode type; here the modal amplitudes  $f_{\rho\eta\bar{p}}$  are vanishingly small in the uniform field limit while  $f_{\bar{p}} = O(1)$ . Thus the solution of Eqs. (3.18) represent a stationary state corresponding to the modal configuration  $A_{\bar{p}}$ .

### 3.2 Discussion of the steady state

Equations (3.18) represent the main results of this section; together with Eqs. (3.9) they

provide a complete specification of the steady state of a laser and its dependence on the many physical and geometrical parameters of the model. We note that Eqs. (3.18) generalize the results of Ref. 24 and establish their domain of validity in more rigorous terms. The steady state formula derived by Lugiato and Milani holds in the uniform field limit for the  $p=0$  mode and resonant operation. In addition, it requires a flat pump profile and a large Fresnel number. To establish the connection between these results in more explicit terms, we note that if the length  $L_A$  of the medium is much smaller than the Rayleigh length  $L\eta_0$ , Eq (3.17a) can be put in the approximate form

$$1 = 2C \frac{4}{\eta_0} \int_0^\infty d\rho \rho \frac{\exp\left[-\frac{\rho^2}{2\rho_p^2} \left(1 + \frac{4\rho_p^2}{\eta_0}\right)\right]}{1 + \frac{4|f_0|^2}{\eta_0} \exp\left(-\frac{2\rho^2}{\eta_0}\right)} \quad (3.19)$$

and if  $\rho_p/\sqrt{\eta_0} \gg 1$ , Eq. (3.19) can be solved explicitly with the result

$$1 = 2C \frac{\eta_0}{4|f_0|^2} \ln \left[ 1 + \frac{4|f_0|^2}{\eta_0} \right] \quad (3.20)$$

This agrees with Eq. (4) of Ref. 24.

The integrals appearing in Eq. (3.18a) cannot be calculated in closed analytic form for arbitrary values of the parameters. It is not difficult, however, to calculate the threshold gain parameter for the  $p=0$  mode using the pump profile (3.5). In fact after setting  $f_0=0$ , Eq. (3.17) leads to

$$(2C)_{\text{thr}} = (1+\Delta^2) \frac{L_A}{L} \left\{ \tau_1 \frac{2\eta_0\psi^2}{\sqrt{1+\psi^2}} \tan^{-1} \left[ \frac{L_A}{2L\eta_0\sqrt{1+\psi^2}} \right] - \tau_2 \frac{L_A}{L} \right\}^{-1} \quad (3.21)$$

where

$$\psi = \frac{2\rho_p}{\sqrt{\eta_0}}$$

Equations (3.21) with  $\tau_1 = 1$  and  $\tau_2=0$  yield the threshold gain related to the pump profile (3.5b); with  $\tau_1 = 2$  and  $\tau_2=1$  we obtain the corresponding result for the pump distribution (3.5c).

An important design parameter is the ratio  $\psi$  between the pump waist  $\rho_p$  and the minimum beam size  $\sqrt{\eta_0}$ . It is clear by inspection that the threshold gain for laser action decreases as  $\psi$  increases, as it should be. The lowest threshold value corresponds to  $\psi \rightarrow \infty$  and is given by the formula

$$(2C)_{\min, \text{thr}} = (1 + \bar{\Delta}^2) \quad (3.22)$$

for both pump models (3.5).

The most obvious difference between the two pump profiles given by Eqs. (3.5) is the behavior of the threshold gain upon varying the parameter  $\psi$ . In the presence of an absorbing region ( $\rho > \rho_p$ ) the threshold gain diverges for a finite critical value  $\psi_c$  such that the denominator in Eq. (3.21) vanishes. If instead the active medium is transparent at the edges of the pumped region, the threshold gain increases monotonically and eventually diverges as  $\psi$  approaches zero. These considerations are summarized in Fig. 2 for typical values of  $\eta_0$  and  $L_A/L$ . [Note that in order to avoid excessive repetitions of the parameters in the figure captions, most relevant values are tabulated in Table 1; additional comments are enclosed in each figure caption, as needed]

The behavior of the modal intensity as a function of the gain parameter allows some interesting observations but shows no major surprises. For example, we looked carefully for the possible presence of bistability near the threshold for laser action, as observed in some earlier experiments<sup>31</sup>, but found no evidence of this behavior. Some typical steady state modal intensities are shown in Figs. 3 and 4 for a resonant and an off-resonant case. It is interesting to note that, for increasing gain parameter, the modal intensity of the radial sidebands can exceed that of the  $p=0$  mode. This effect is related to the different transverse profile of the various radial eigenfunctions



and therefore is strictly a transverse effect. When the pumped region is sufficiently broader than the waist of the  $TEM_{00}$  mode, higher order modes can take better advantage of the available gain because of their greater modal extent (see Fig. 3). The effect is enhanced in the presence of a detuning (Fig. 4) because of the increased coupling between the radial sidemodes and the atomic line. Figures 3 and 4 correspond to a pump profile of the type given by Eq. (3.5b) with a parameter  $\psi = 10$  (i.e. with a large pump waist relative to the beam size). In this case one expects that the pump profile (3.5c) should yield rather similar results. Indeed the differences are small and only quantitative in character.

We also expect that the shape of the pump profile should play a more significant role when the ratio  $2\rho_p/\sqrt{\eta_0}$  is closer to unity. A comparison between the steady state curves obtained from Eq. (3.5b) with those resulting from Eq. (3.5c) is shown in Fig. 5. Here again the result is not surprising if we consider that, in the latter case, the presence of an absorbing ring around the edges of the medium is bound to affect the higher order radial modes more than it affects the  $TEM_{00}$  configuration.

It is also interesting to note that the crossing of the modal intensity curves is a transverse effect which is less pronounced when the intensity profile of the beam becomes more homogeneous in the radial direction, i.e. when  $\rho_p/\sqrt{\eta_0} \gg 1$ . An example is shown in Fig. 6 for a pump profile of the type (3.5b).

The dependence of the modal intensity on the detuning parameter  $\bar{\delta}_{AC}$  is shown in Fig. 7 for a set of parameters where more than one transverse mode can be above threshold for the same value of  $\bar{\delta}_{AC}$ . In general, this result does not imply that more than one mode will be above threshold at the same time because the presence of a given steady state alters the unsaturated gain distribution. Based on past experience with the plane wave model<sup>6g</sup> we expect that modes  $p=0$  and  $p=1$ , under the conditions of the figure, will compete and produce either discontinuous jumps (mode-hopping) or undamped pulsations through a mechanism that was identified as a phase instability in the plane-wave model. Naturally, this is a conjecture at this level; the issue can only be verified in the

context of the linear stability analysis, as we shall discuss in the next section.

We conclude this section by noting the qualitatively similar behavior of the output intensity of a CO<sub>2</sub> laser during a detuning scan, as shown in Fig. 8 .

#### 4. Linear stability analysis

The starting point of our analysis is the set of equations (3.8) for the modal amplitudes and for the atomic variables, together with the boundary conditions for the ring resonator. With the laser in a steady state we ask whether fluctuations corresponding to the modal indices (n,p) can grow exponentially out of noise. This is the usual picture behind the onset of unstable behavior. The real part of the rate constant for this process yields the rate of growth of the unstable modes, while the imaginary part measures the oscillation frequency resulting from the beat of the unstable mode and the steady state field.

As our first step we map the coordinates  $\eta$  and  $\tau$  into the new pair of independent variables  $\eta'$  and  $\tau'$  according to the transformation

$$\begin{aligned}\eta' &= \eta \\ \tau' &= \tau + \gamma_{\perp} \frac{\Lambda-L}{c} (\eta+1/2)\end{aligned}\quad (4.1)$$

with the result that the new boundary conditions take on an isochronous form. Next, we linearize the equations of motion (3.8) and obtain

$$\frac{\partial \delta f_p}{\partial \eta'} + \frac{1}{v} \frac{\partial \delta f_p}{\partial \tau'} = i \frac{\bar{\omega}}{v} f_p - \alpha L \int_0^{\infty} d\rho \rho A_p^*(\rho, \eta) \delta \bar{P} \quad (4.2a)$$

$$\frac{\partial \delta \bar{P}}{\partial \tau'} = -\{\delta \bar{F} D_{st} + \bar{F}_{st} \delta D + (1+i\bar{\Delta}) \delta \bar{P}\} \quad (4.2b)$$

$$\frac{\partial \delta D}{\partial \tau'} = -\bar{\gamma} \left\{ -\frac{1}{2} (\delta \bar{F}^* \bar{P}_{st} + \bar{F}_{st}^* \delta \bar{P} + \delta \bar{F} \bar{P}_{st}^* + \bar{F}_{st} \delta \bar{P}^*) + \delta D \right\} \quad (4.2c)$$

where

$$\delta \bar{F}(\rho, \eta', \tau') = \sum_p A_p(\rho, \eta') \delta f_p(\eta', \tau') \quad (4.3)$$

In addition, of course, one must also include linearized equations for  $\delta f_p^*$  and  $\delta \bar{P}^*$  which are complex conjugate of Eqs. (4.2a) and (4.2b), respectively. The boundary conditions are

$$\delta f_p(-1/2, \tau') = R e^{-i \delta_p} e^{i \bar{\delta} \Omega \gamma_L \frac{\Lambda-L}{c}} \delta f_p(1/2, \tau') \quad (4.4)$$

as one can verify at once using Eq. (3.8d) in the new reference frame.

Following the standard procedure we introduce the ansatz

$$\begin{pmatrix} \delta f_p(\eta', \tau') \\ \delta f_p^*(\eta', \tau') \\ \delta \bar{P}(\rho, \eta', \tau') \\ \delta \bar{P}^*(\rho, \eta', \tau') \\ \delta D(\rho, \eta', \tau') \end{pmatrix} = e^{\bar{\lambda} \tau'} \begin{pmatrix} \delta \varphi_p(\eta') \\ \delta \varphi_p^*(\eta') \\ \delta p(\rho, \eta') \\ \delta p^*(\rho, \eta') \\ \delta d(\rho, \eta') \end{pmatrix} \quad (4.5)$$

in the linearized equations (4.2);  $\bar{\lambda}$  denotes the linearized rate constant in units of  $\gamma_L$ . The equations for the atomic fluctuation variables are algebraic in nature and can be solved at once with the result

$$\begin{aligned} \delta p(\rho, \eta') &= T_1(\rho, \eta', \bar{\lambda}) \sum_{p'} A_{p'}(\rho, \eta') \delta \varphi_{p'}(\eta') + \\ &\quad T_2(\rho, \eta', \bar{\lambda}) \sum_{p'} A_{p'}^*(\rho, \eta') \delta \varphi_{p'}^*(\eta') \end{aligned} \quad (4.6a)$$

$$\begin{aligned} \delta p^*(\rho, \eta') &= T_2^*(\rho, \eta', \bar{\lambda}) \sum_{p'} A_{p'}(\rho, \eta') \delta \varphi_{p'}(\eta') + \\ &\quad T_1^*(\rho, \eta', \bar{\lambda}) \sum_{p'} A_{p'}^*(\rho, \eta') \delta \varphi_{p'}^*(\eta') \end{aligned} \quad (4.6b)$$

where

$$T_1 = \frac{c_3 c_1^* - c_2 c_4^*}{c_1 c_1^* - c_2 c_2^*} \quad (4.7a)$$

$$T_2 = \frac{c_4 c_1^* - c_2 c_3^*}{c_1 c_1^* - c_2 c_2^*} \quad (4.7b)$$

and

$$c_1(\rho, \eta') = \bar{\lambda} + 1 + i\bar{\Delta} + \frac{1}{2} \frac{\bar{\gamma}}{\bar{\lambda} + \bar{\gamma}} |\bar{F}_{st}|^2 \quad (4.8a)$$

$$c_2(\rho, \eta') = \frac{1}{2} \frac{\bar{\gamma}}{\bar{\lambda} + \bar{\gamma}} \bar{F}_{st}^2 \quad (4.8b)$$

$$c_3(\rho, \eta') = -(D_{st} + \frac{1}{2} \frac{\bar{\gamma}}{\bar{\lambda} + \bar{\gamma}} \bar{F}_{st} \bar{P}_{st}^*) \quad (4.8c)$$

$$c_4(\rho, \eta') = -\frac{1}{2} \frac{\bar{\gamma}}{\bar{\lambda} + \bar{\gamma}} \bar{F}_{st} \bar{P}_{st} \quad (4.8d)$$

The symbols  $c_1^*$ ,  $c_2^*$  etc. indicate the complex conjugate parameters of  $c_1$  and  $c_2$ , etc.; we note, however, that in performing the complex conjugate operation we must handle  $\bar{\lambda}$  as if it were a real variable, i.e.  $c_1^*$  is a function of  $\bar{\lambda}$  and not  $\bar{\lambda}^*$ . The reason for this peculiar rule can be traced back easily to the equations of motion for  $\delta p$  and  $\delta p^*$  and to the ansatz (4.5).

At this point the linearized field equations become

$$\begin{aligned} \frac{\partial}{\partial \eta'} \delta \varphi_p + \gamma_{\perp} \frac{\Delta}{c} \bar{\lambda} \delta \varphi_p = i \frac{\bar{\delta} \Omega}{v} \delta \varphi_p - \alpha L \int_0^{\infty} d\rho \rho A_p^*(\rho, \eta') \{ T_1(\rho, \eta', \bar{\lambda}) \times \\ \sum_{p'} A_p(\rho, \eta') \delta \varphi_{p'}(\eta') + T_2(\rho, \eta', \bar{\lambda}) \sum_{p'} A_p^*(\rho, \eta') \delta \varphi_{p'}^*(\eta') \} \end{aligned} \quad (4.9a)$$

$$\begin{aligned} \frac{\partial}{\partial \eta'} \delta \varphi_p^* + \gamma_{\perp} \frac{\Delta}{c} \bar{\lambda} \delta \varphi_p^* = -i \frac{\bar{\delta} \Omega}{v} \delta \varphi_p^* - \alpha L \int_0^{\infty} d\rho \rho A_p(\rho, \eta') \{ T_2^*(\rho, \eta', \bar{\lambda}) \times \\ \sum_{p'} A_p(\rho, \eta') \delta \varphi_{p'}(\eta') + T_1^*(\rho, \eta', \bar{\lambda}) \sum_{p'} A_p^*(\rho, \eta') \delta \varphi_{p'}^*(\eta') \} \end{aligned} \quad (4.9b)$$

The appropriate boundary conditions take the form

$$\delta\phi_p(-1/2) = R e^{-i\bar{\delta}_p} \delta\phi_p(1/2) \quad (4.9c)$$

where

$$\bar{\delta}_p = \delta_p - \bar{\delta}\Omega \gamma_{\perp} \frac{\Lambda-L}{c} \quad (4.10)$$

Before taking the final steps with the calculation of the linearized eigenvalues it is convenient to introduce one more change of dependent variables whose main purpose is to transform the boundary conditions to a standard periodicity form. This can be accomplished with the definitions

$$\delta\phi_p(\eta') = v_p(\eta') e^{-(\eta'+1/2)(\ln R - i\bar{\delta}_p)} \quad (4.11a)$$

$$\delta\phi_p^*(\eta') = v_p^*(\eta') e^{-(\eta'+1/2)(\ln R + i\bar{\delta}_p)} \quad (4.11b)$$

because, as we can easily see with the help of Eqs. (4.9c) and (4.11), we have

$$v_p(-1/2) = v_p(1/2) \quad (4.12a)$$

$$v_p^*(-1/2) = v_p^*(1/2) \quad (4.12b)$$

The transformed field fluctuation equations take the form

$$\begin{aligned} \frac{c}{\Lambda\gamma_{\perp}} \frac{\partial v_p}{\partial \eta'} + \left( \bar{\kappa} + i \frac{c}{\Lambda\gamma_{\perp}} \delta_p - i \bar{\delta}\Omega \right) v_p + \bar{\lambda} v_p = - \bar{\kappa} 2C \frac{L}{L_A} \int_0^{\infty} d\rho \rho A_p^*(\rho, \eta') \{ \\ T_1(\rho, \eta', \bar{\lambda}) \sum_{p'} A_{p'}(\rho, \eta') v_{p'}(\eta') e^{i(\eta'+1/2)(\bar{\delta}_{p'} - \bar{\delta}_p)} + \\ T_2(\rho, \eta', \bar{\lambda}) \sum_{p'} A_{p'}^*(\rho, \eta') v_{p'}(\eta') e^{-i(\eta'+1/2)(\bar{\delta}_{p'} + \bar{\delta}_p)} \} \end{aligned} \quad (4.13)$$

with a similar equation for  $v_p^*$  following from the complex conjugation of (4.13).

The required linearized eigenvalues cannot be calculated in closed form for arbitrary values of the parameters in spite of the linear nature of Eq. (4.13) and its complex conjugate. However, in the good cavity limit,  $\bar{\kappa} \ll 1$ , it is possible to develop a perturbative procedure whose end result is the analytic derivation of the instability condition. Numerical calculations are needed only to carry

out the quadrature of the final integral expression. Our strategy can be summarized as follows: by taking advantage of the periodic nature of the boundary conditions (4.12), we expand  $v_p$  and  $v_p^*$  in appropriate Fourier series over the interval  $-1/2 \leq \eta \leq 1/2$  according to the usual relations

$$v_p = \sum_n e^{ik_n L \eta} \sigma_{n,p} \quad (4.14a)$$

$$v_p^* = \sum_n e^{-ik_n L \eta} \sigma_{n,p}^* \quad (4.14b)$$

where  $k_n = 2\pi n/L$  ( $n = 0, \pm 1, \pm 2, \dots$ ) and  $\sigma_{n,p}$  and  $\sigma_{n,p}^*$  are complex coefficients; next we construct an algebraic system of equations for the fluctuation amplitudes  $\sigma_{n,p}$  and  $\sigma_{n,p}^*$  with the help of the orthonormality relation

$$\int_{-1/2}^{1/2} d\eta' e^{ik_n L \eta'} e^{-ik_m L \eta'} = \delta_{n,m} \quad (4.15)$$

Finally, we only need to solve the resulting linear algebraic problem to first order in  $\bar{\kappa}$ .

Application of Eqs. (4.14) and (4.15) leads to

$$i \bar{\alpha}_n \sigma_{n,p} + (\bar{\kappa} + i \frac{c}{\Lambda \gamma_{\perp}} \delta_p - i \bar{\delta} \Omega) \sigma_{n,p} + \bar{\lambda} \sigma_{n,p} = \bar{\kappa} \sum_{n',p'} (\Phi_{n,p}^{n',p'}(\bar{\lambda}) \sigma_{n',p'} + \Psi_{n,p}^{n',p'}(\bar{\lambda}) \sigma_{n',p'}^*) \quad (4.16a)$$

$$-i \bar{\alpha}_n \sigma_{n,p}^* + (\bar{\kappa} - i \frac{c}{\Lambda \gamma_{\perp}} \delta_p + i \bar{\delta} \Omega) \sigma_{n,p}^* + \bar{\lambda} \sigma_{n,p}^* = \bar{\kappa} \sum_{n',p'} (\Phi_{n,p}^{n',p'}(\bar{\lambda})^* \sigma_{n',p'}^* + \Psi_{n,p}^{n',p'}(\bar{\lambda})^* \sigma_{n',p'}) \quad (4.16b)$$

where

$$\bar{\alpha}_n = n \bar{\alpha}_1 = \frac{2\pi n c}{\Lambda \gamma_{\perp}}$$

is the frequency separation between the  $n$ -th longitudinal resonance and the reference frequency measured in units of  $\gamma_{\perp}$ , and the coefficients  $\Phi$  and  $\Psi$  are defined by

$$\Phi_{n,p}^{n',p'}(\bar{\lambda}) = -2C \frac{L}{L_A} \int_{-\frac{L_A}{2L}}^{+\frac{L_A}{2L}} d\eta' e^{i(k_n - k_n)L\eta'} e^{i(\eta'+1/2)(\bar{\delta}_p - \bar{\delta}_p)} \int_0^\infty d\rho \rho A_p^*(\rho, \eta') \times$$

$$A_p(\rho, \eta') T_1(\rho, \eta', \bar{\lambda}) \quad (4.17a)$$

$$\Psi_{n,p}^{n',p'}(\bar{\lambda}) = -2C \frac{L}{L_A} \int_{-\frac{L_A}{2L}}^{+\frac{L_A}{2L}} d\eta' e^{-i(k_n + k_n)L\eta'} e^{i(\eta'+1/2)(\bar{\delta}_p + \bar{\delta}_p)} \int_0^\infty d\rho \rho A_p^*(\rho, \eta') \times$$

$$A_p^*(\rho, \eta') T_2(\rho, \eta', \bar{\lambda}) \quad (4.17b)$$

The solution of Eqs. (4.16) is equivalent to the diagonalization of the problem

$$L \vec{v} = \bar{\lambda} \vec{v} \quad (4.18)$$

where the matrix  $L$ , whose definition is obvious from Eqs. (4.16), has the structure

$$L = L_0 + \bar{\kappa} L_1 \quad (4.19)$$

and  $\vec{v}$  is the column vector with components  $\sigma_{n,p}$  and  $\sigma_{n,p}^*$ . Now we assume the good cavity condition  $\bar{\kappa} \ll 1$  and proceed to evaluate the eigenvalues to first order in  $\bar{\kappa}$ . We assume also that the longitudinal mode spacing  $\alpha_1 = 2\pi c/\Lambda$  and the atomic linewidth  $\gamma_\perp$  are of the same order of magnitude. Hence, the good cavity condition

$$\bar{\kappa} = \left( \frac{\alpha_1}{2\pi\gamma_\perp} \right) T \ll 1$$

is equivalent to the uniform field condition  $T \ll 1$ .

We consider now the stationary solution

$$\bar{F}_{st}(\rho, \eta') = A_0(\rho, \eta') f_0(\eta') \quad (4.20)$$

for which  $\bar{\delta}\Omega$  is of the order of  $\bar{\kappa}$ .

(a) Consider first the case  $p \neq 0$ . We select two values of the indices  $n$  and  $p$  (say,  $\bar{n}$  and  $\bar{p}$ )

and assume that there be no accidental degeneracy of the type

$$\bar{\alpha}_n + \frac{c}{\Lambda \gamma_{\perp}} \delta_p = \pm (\bar{\alpha}_n + \frac{c}{\Lambda \gamma_{\perp}} \delta_p) \quad (4.21)$$

for any  $n$  and  $p$  other than the chosen  $\bar{n}$  and  $\bar{p}$ . Under this condition the eigenstates of  $L$  have the form

$$\sigma_{\bar{n}, \bar{p}} = \sigma_{\bar{n}, \bar{p}}^{(0)} + O(\bar{\kappa}) = \sigma_{\bar{n}, \bar{p}}^{(0)} + \bar{\kappa} \sigma_{\bar{n}, \bar{p}}^{(1)} \quad (4.22a)$$

and

$$\sigma_{n, p} = O(\bar{\kappa}) \quad (4.22b)$$

$$\sigma_{n, p}^* = O(\bar{\kappa}) \quad (4.22c)$$

Equations (4.22b,c) apply to  $n, p \neq \bar{n}, \bar{p}$ . Consider now Eq (4.16a) with  $n, p = \bar{n}, \bar{p}$  and let

$$\bar{\lambda} = \bar{\lambda}^{(0)} + \bar{\kappa} \lambda^{(1)} = -i \left( \bar{\alpha}_{\bar{n}} + \frac{c}{\Lambda \gamma_{\perp}} \delta_{\bar{p}} \right) + \bar{\kappa} \lambda^{(1)} \quad (4.23)$$

After substitution of Eqs. (4.22) and (4.23) into Eq. (4.16a) we obtain

$$-i \left( \bar{\alpha}_{\bar{n}} + \frac{c}{\Lambda \gamma_{\perp}} \delta_{\bar{p}} \right) \sigma_{\bar{n}, \bar{p}}^{(0)} = \bar{\lambda}^{(0)} \sigma_{\bar{n}, \bar{p}}^{(0)} \quad (4.24a)$$

and

$$\lambda^{(1)} = -1 + i \frac{\bar{\delta} \Omega}{\bar{\kappa}} + \Phi_{\bar{n}, \bar{p}}^{\bar{n}, \bar{p}} (\bar{\lambda}^{(0)}) \quad (4.24b)$$

where

$$\Phi_{\bar{n}, \bar{p}}^{\bar{n}, \bar{p}} (\bar{\lambda}^{(0)}) = -2C \frac{L}{L_A} \int_{-\frac{L_A}{2L}}^{+\frac{L_A}{2L}} d\eta' \int_0^{\infty} d\rho \rho |A_{\bar{p}}(\rho, \eta')|^2 T_1(\rho, \eta', \bar{\lambda}^{(0)}) \quad (4.24c)$$

Equation (4.24a) gives the non-degenerate lowest order eigenvalues, while (4.24b) gives their first order corrections.



(b) Consider now the case  $p=0$ . We select a fixed value of  $n=\bar{n}$  and note that the eigenvalue

$$\bar{\lambda}^{(0)} = -i \bar{\alpha}_{\bar{n}}$$

is doubly degenerate, as we can easily confirm by considering Eq. (4.16a) for  $\sigma_{\bar{n},0}$  and Eq. (4.16b) for  $\sigma_{-\bar{n},0}^*$ . It follows that the corresponding eigenvalues are of the form

$$\sigma_{\bar{n},0} = \sigma_{\bar{n},0}^{(0)} + O(\bar{\kappa}) \quad (4.25a)$$

$$\sigma_{-\bar{n},0}^* = (\sigma_{-\bar{n},0}^{(0)})^* + O(\bar{\kappa}) \quad (4.25b)$$

$$\sigma_{n,p} = O(\bar{\kappa}) \quad \text{for } n,p \neq \bar{n},0 \quad (4.25c)$$

$$\sigma_{n,p}^* = O(\bar{\kappa}) \quad \text{for } n,p \neq -\bar{n},0 \quad (4.25d)$$

If we now set  $(n,p) = (\bar{n},0)$  in Eq. (4.16a), and  $(n,p) = (-\bar{n},0)$  in Eq. (4.16b), and equate the first order terms in  $\bar{\kappa}$  we obtain

$$(1+\lambda^{(1)}) \sigma_{\bar{n},0}^{(0)} = \Phi_{\bar{n},0}^{\bar{n},0}(\bar{\lambda}^{(0)}) \sigma_{\bar{n},0}^{(0)} + \Psi_{-\bar{n},0}^{\bar{n},0}(\bar{\lambda}^{(0)}) \sigma_{-\bar{n},0}^{*(0)} \quad (4.26a)$$

$$(1+\lambda^{(1)}) \sigma_{-\bar{n},0}^{*(0)} = \Phi_{-\bar{n},0}^{*\bar{n},0}(\bar{\lambda}^{(0)}) \sigma_{-\bar{n},0}^{*(0)} + \Psi_{\bar{n},0}^{\bar{n},0}(\bar{\lambda}^{(0)}) \sigma_{\bar{n},0}^{(0)} \quad (4.26b)$$

These equations lead to the first order corrections  $\lambda^{(1)}$  of the eigenvalues in the degenerate case. In particular, in the limit  $L_A \ll L\eta_0$  and in resonance ( $\bar{\delta}_{AC} = 0$ ) we find the results already obtained by Lugiato and Milani<sup>24</sup>.

If the stationary state is of the type

$$\bar{F}_{st}(\rho, \eta') = A_q(\rho, \eta') f_q(\eta') \quad (4.27)$$

corresponding to a single radial mode  $q$ , the above treatment needs only a few simple modifications. Equations (4.16) still hold with  $\bar{\delta}\Omega$  replaced by  $\bar{\delta}\Omega_q$ . The frequency offset  $\bar{\delta}\Omega_q$  given by Eq. (18b) contains a large contribution and a correction of order  $\bar{\kappa}$ .

The unperturbed eigenvalues [Eq. (4.24a)] contain only the large contributions and are given by

$$\bar{\lambda}_{n,p}^{(0)} = -i \left( \bar{\alpha}_n + \frac{c}{\Lambda \gamma_{\perp}} (\delta_p - \delta_q) \right) \quad (4.28)$$

and of course the symbol  $\bar{\lambda}^{(0)}$  that appears in Eqs (4.24b) and (4.26) must be interpreted according to Eq. (4.28).

Now we consider the linearized stability problem in the full adiabatic elimination regime of the atomic variables (i.e. when  $\gamma_{\perp}, \gamma_{\parallel} \gg c/\Lambda$ ). Here again the general procedure needs only a few minor modifications in order to fit the limiting situation. In fact, with reference to Eqs (4.2b) and (4.2c) the adiabatic elimination limit corresponds to setting both time derivatives equal to zero. In this case the resulting expressions given by Eqs. (4.8) remain valid provided we set  $\lambda=0$ . Thus the linearized eigenvalues in this limit are still given by Eqs. (4.24b) and (4.26) with  $\lambda^{(0)}$  formally equal to zero and  $T_1$  and  $T_2$  given by the simple formulas

$$T_1(\rho, \eta) = -\bar{\chi}(\rho) \frac{(1+\bar{\Delta}^2)(1-i\bar{\Delta})}{(1+\bar{\Delta}^2+|\bar{F}_{st}|^2)^2} \quad (4.29a)$$

$$T_2(\rho, \eta) = \bar{\chi}(\rho) \frac{(1-i\bar{\Delta})\bar{F}_{st}^2}{(1+\bar{\Delta}^2+|\bar{F}_{st}|^2)^2} \quad (4.29b)$$

In this case  $\lambda^{(1)}$  depends on  $\bar{p}$  only by way of the modal function  $|A_{\bar{p}}|^2$  while, away from the adiabatic regime, it depends on the radial index also through the unperturbed frequency

$$\bar{\lambda}_{\bar{p}}^{(0)} = -i \left[ \bar{\alpha}_{\bar{n}} + \frac{c}{\Lambda \gamma_{\perp}} \delta_{\bar{p}} \right]$$

Another interesting limit is one in which the length  $L_A$  of the atomic sample is much smaller than the Rayleigh length  $L\eta_0$ . In this case Eq. (4.24c) reduces to

$$\Phi_{\bar{n}, \bar{p}}^{\bar{n}, \bar{p}}(\bar{\lambda}^{(0)}) = -2C \frac{4}{\eta_0} \int_0^{\infty} d\rho \rho \exp\left(-\frac{2\rho^2}{\eta_0}\right) L_{\bar{p}}^2\left(\frac{2\rho^2}{\eta_0}\right) T_1(\rho, \eta', \bar{\lambda}^{(0)}) \quad (4.30)$$

and in the expression for  $T_1$  given by Eqs. (4.7a) and (4.8), one must replace  $|\bar{F}_{st}|$  with the

expression

$$|\bar{F}_{st}|^2 = \frac{4}{\eta_0} \exp\left(-\frac{2\rho^2}{\eta_0}\right) L_q^2\left(\frac{2\rho^2}{\eta_0}\right) f_q \quad (4.31)$$

It is important to remark that the gain parameter  $C$  in Eq. (4.24c) is related to the amplitude  $f_{\bar{p}}$  of the stationary field, which appears in  $T_1$  through  $\bar{F}_{st}$ , by the state equation (3.18). In addition, we note that, for correctness, the expression for  $\bar{\Delta}$  which is needed in calculating  $T_1$  should be evaluated to order zero in  $\bar{\kappa}$ . Finally, because  $\bar{\lambda}^{(0)}$  is a pure imaginary number, the instability condition is given by the equation

$$\text{Re } \lambda^{(1)} > 0 \quad (4.32)$$

## 5. Stability analysis: Numerical results

The main results of the previous section can be summarized as follows:

- i) In the extended uniform field limit (3.12a) and (3.12b) with  $\alpha_1$  and  $\gamma_{\perp}$  of the same order of magnitude, a single-mode steady state field characterized by modal indices  $\bar{n}$  and  $\bar{p}$  becomes unstable if

$$\text{Re } \lambda_{n,p}^{(1)} > 0 \quad (5.1)$$

for at least a pair of values of the indices  $n$  and  $p$ .

- ii) If  $p \neq \bar{p}$ , Eq. (5.1) takes the explicit form

$$-1 + \text{Re } \Phi_{n,p}^{n,p}(\bar{\lambda}^{(0)}) > 1 \quad (5.2)$$

where

$$\Phi_{n,p}^{n,p}(\bar{\lambda}^{(0)}) = -2C \frac{L}{L_A} \int_{-\frac{L_A}{2L}}^{\frac{L_A}{2L}} d\eta' \int_0^{\infty} d\rho \rho |A_p(\rho, \eta')|^2 T_1(\rho, \eta', \bar{\lambda}^{(0)}) \quad (5.3)$$

$$\bar{\lambda}^{(0)} = -i \left( \bar{\alpha}_n + \frac{c}{\Lambda \gamma_{\perp}} \delta_p \right) \quad (5.4)$$

and the function  $T_1$  is defined by Eqs. (4.7) and (4.8).

- iii) If  $p = \bar{p}$ ,  $\lambda^{(1)}$  is the solution of the quadratic equation that results from the linear homogeneous system (4.25).
- iv) The instability threshold for the mode with indices  $n$  and  $p$ , also known as the second laser threshold, is the value of  $2C$  that satisfies the identity

$$-1 + \text{Re } \Phi_{n,p}^{n,p}(\bar{\lambda}^{(0)}) = 1 \quad (5.5)$$

To avoid confusion between this threshold value and the threshold for ordinary laser action, we denote the former by  $2C^{(2)}$  and the latter by  $2C^{(1)}$ .

The parameter space is much wider than in the case of the usual plane-wave theory. Numerous geometrical and physical parameters play an important role in providing the conditions for unstable behavior so that a complete scan of the relevant space is a monumental task. Here, in an attempt to aid our analysis and to provide some guidelines for future searches, we have identified the parameters that can be viewed as independent of one another, at least from a theoretical standpoint. For the general detuned case, i.e. when  $\bar{\delta}_{AB} \neq 0$ , these are listed in Table 2. In the resonant case, and with the additional restriction that the steady state field is of the  $TEM_{00}$  type, the independent parameters are  $2C$ ,  $\eta_0$ ,  $\psi = \rho_p / \sqrt{\eta_0}$ ,  $\bar{\delta}\omega_1$ ,  $\bar{\gamma}$  and  $L_A/L$ ; note that  $\bar{\delta}\omega_1 = \delta_1 \bar{\alpha}_1 / 2\pi$ .

The numerical results collected in this section represent a typical cross section of data that we have calculated in a fairly large size (but definitely not exhaustive) scan of the parameter space. Undoubtedly, additional useful facts will emerge from future studies. For convenience we have divided our survey into two subsections. The first relates to a resonant laser ( $\bar{\delta}_{AC} = 0$ ); the second to a detuned situation.

#### A) Resonant lasers

The instability threshold is rather sensitive to the cavity geometry. In fact, the second laser threshold is a monotonic function of the transverse mode spacing, as shown in Fig. 9 where we display this dependence for three values of  $\bar{\gamma}$ . On the basis of this result one can infer that long resonators or cavities whose curved mirrors have large radii of curvature should be more sensitive to instabilities even in the vicinity of the threshold for laser action. In addition, active media with a slow rate of population decay, relative to the decay of the polarization, are more unstable than those with comparable rates of decay for the population and the polarization. This behavior is quite different from what is predicted by the plane-wave theory where the axial mode spacing has very little influence on the value of the instability threshold<sup>6e</sup>.

As expected, if the laser operates in a  $TEM_{00}$  configuration the first radial sideband ( $p=1$ ) is more likely to become unstable than the higher order radial sidemodes ( $p=2, 3, \dots$ ); this is illustrated in Fig. 10. The reduced influence of the higher order transverse modes on the unstable dynamics is linked to their progressively larger cross section which causes the cavity field to see a smaller unsaturated gain because the equilibrium population difference  $\bar{\chi}$  tapers radially out in a pre-assigned way. Furthermore, in practical devices, intracavity apertures provide an even more stringent selection mechanism by increasing the losses experienced by the higher order modes to the point that they can no longer meet the ordinary threshold condition.

One of the key parameters that controls the appearance of low threshold instabilities is the ratio between the transverse size of the active volume and the beam cross section. Figure 11 shows the large drop of the instability threshold that accompanies an increase in the parameter  $\psi$  for both modes  $p=1$  and  $p=2$ . Figure 11a corresponds to the pump model (3.5b), while Fig. 11b corresponds to (3.5c). This behavior is consistent with the results already shown in the previous figures so that again, for example, mode  $p=2$  tends to be more stable than mode  $p=1$ . Furthermore, the presence of an absorbing region at the radial edges of the active medium (Fig. 11b) causes a lowering of the effective ratio between the transverse size of the gain volume and the beam waist; hence, the stability of the system is enhanced in the case shown in Fig. 11b relative to

that of Fig. 11a.

### B) Detuned laser

The effect of detuning is rather interesting also, although, as indicated in Table 1, a complete survey of the various possibilities will require a much more extensive study than we have carried out thus far. Figure 12 contains a composite map of the instability threshold values plotted as functions of the detuning parameter  $\bar{\delta}_{AC}$  for four different off-resonant modes, under the assumption that the steady state of the laser is characterized by the modal indices  $(\bar{n}, 0)$ . The transverse modes  $(\bar{n}, 1)$  and  $(\bar{n}, 2)$  exhibit the same relative stability properties as displayed in previous figures, with mode  $(\bar{n}, 2)$  always being more stable than mode  $(\bar{n}, 1)$ . This figure shows also that transverse modes belonging to the nearest longitudinal mode  $(\bar{n}-1)$  can play an important role in the dynamics of this system. Their resonances are shown along the horizontal frequency axis of the graph by vertical bars which indicate that modes  $(\bar{n}-1, 1)$  and  $(\bar{n}-1, 2)$  are clustered around the resonant mode and can easily become unstable as we vary the detuning parameter. Mode  $(\bar{n}-1, 2)$ , in particular, happens to fall in the vicinity of the resonant mode and has a low instability threshold for a fairly wide range of the detuning parameter. We recall, however, that in real laser systems the losses of higher order transverse modes are higher than those of lower order modes, so that, in practice, the resonances characterized by an index  $p$  equal to or greater than 2 are likely to have no significant dynamical influence.

In an earlier paper dealing with the role of the Gaussian transverse profile on the dynamics of a ring laser, Lugiato and Milani<sup>24</sup> proposed the surprising result that the Risken-Nummedal<sup>6b,c</sup> instability disappears if every field mode has a  $TEM_{00}$  structure. Their calculation envisioned a ring laser operating in resonance and explored the possible existence of unstable axial modes with a different value of the longitudinal index. After assuming a flat pump profile (this gives the best chance for the development of an instability, as shown by our Fig. 11) they concluded that no longitudinal sidebands would ever be unstable.

If we consider that the theoretical threshold values for a Risken-Nummedal plane-wave instability are already unreasonably high, this result is very surprising because it suggests that transverse effects have a stabilizing action on laser dynamics. We confirm the results of Ref. 24 in Fig. 13. In fact, on resonance ( $\bar{\delta}_{AC}=0$ ), the sidemode  $(\bar{n}+1, 0)$  is indeed stable for all values of the gain (the same conclusion holds for all other  $TEM_{00}$  modes). It becomes unstable, instead, for sufficiently large values of the detuning parameter, as anticipated in Ref. 32, following a pattern that is typical of the so-called phase instability of the plane-wave theory<sup>6g</sup>. In the context of our formulation we cannot introduce a clear-cut discrimination between instabilities of the amplitude and phase type because of the nature of the linearized spectrum of eigenvalues<sup>33</sup>. However, the behavior displayed in Fig. 13 is very reminiscent of the latter type of instability.

The combined message of our results is that the transverse profile of the field and of the atomic variables does play an important role in the emergence of low threshold instabilities, but the radial modes with the same longitudinal label as the operating steady state of the laser are the ones that are most closely responsible for introducing output pulsations.

## 6. Conclusion and comments

With a suitable extension of the uniform field limit [Eq. (3.12)] we have succeeded in carrying out a detailed analytical study of the steady state and linear stability properties of a ring laser containing a homogeneously broadened active medium. In this limited context we have been able to incorporate the effects of curved reflecting surfaces and the possible lack of transverse uniformity of the pump. The main claim of this paper is that these transverse effects can account, at least in good measure, for the emergence of low threshold unstable behavior.

We have identified the most critical parameters that affect the stability of this system. According to our results, instabilities are enhanced by a flat pump profile relative to the transverse shape of the field. To be more precise we found that if the ratio  $2\rho_P/\sqrt{\eta_0}$  becomes sufficiently

large, the threshold for instability decreases dramatically and can be only a few percent higher than the ordinary laser threshold. This is probably one of the most specific signs of the transverse instability. In fact, longitudinal instabilities are favored in the limiting configuration  $\rho_P/\sqrt{\eta_0} \rightarrow 0$  with very large values of the gain parameter  $C^{17e,24}$ , while the transverse instabilities acquire a dominant role in the opposite limit ( $\rho_P/\sqrt{\eta_0} \rightarrow \infty$  with a small value of  $C$ ). In this sense, we can distinguish between longitudinal and transverse instabilities. The former type is, of course, a high threshold phenomenon.

As already mentioned, each transverse mode with  $p \neq 0$  is associated with only one eigenvalue, in contrast with the situation that prevails in the plane-wave theory where two linearized eigenvalues exist for each value of the longitudinal index  $n$ . This precludes a straightforward distinction between amplitude and phase instabilities. The obvious cause is that the longitudinal modes are distributed symmetrically on both sides of the resonant mode, while the transverse resonances ( $p=1,2,\dots$ ) lie only on one side of it.

Another significant difference between the present theory and its plane-wave counterpart is the basic role played by the modal functions in assigning a different spatial structure to the possible field configurations [compare the  $A_p(\rho, \eta)$  modes with the functions  $\exp(ik_n L \eta)$ ]. One of the immediate consequences of this fact is that all the field fluctuation variables are coupled to one another to form an infinite system of linear equations [Eqs. (4.9)]. In contrast, the corresponding fluctuation equations for the plane-wave theory<sup>6g</sup> break-up into an infinite set of  $(2 \times 2)$  blocks.

An important and fundamental difference between our model and earlier plane-wave analyses is that our calculations predict the existence of unstable behaviors in the rate equation regime ( $\partial P/\partial t = 0$ ) and even in the limit when both atomic variables are eliminated adiabatically ( $\partial P/\partial t = \partial D/\partial t = 0$ ). This result stands in striking contrast with the well known behavior of the plane-wave Maxwell-Bloch equations which are always stable in the rate equation and in the full adiabatic elimination regime.

It is interesting to compare the setting and the results of this paper with a recent analysis by



Lugiato and Lefever<sup>34</sup> dealing with the spontaneous formation of space instabilities. An important difference is that the low threshold temporal pulsations described in this paper require a radial intermode spacing  $\delta\omega_1$  of the order of the longitudinal mode spacing [i.e.  $\bar{\delta\omega}_1 = O(1)$ ]. The emergence of spatial instabilities requires, instead, that the transverse modes have overlapping profiles in frequency and this implies that  $\bar{\delta\omega}_1 = O(\bar{\kappa})$ .

In a sense this paper is complementary to the one cited in Ref. 34; of course, the medium is active in our case and not driven by an injected signal, in contrast to the situation described in Ref. 34; furthermore the cavity geometry is of the ring type with two spherical mirrors, while the resonator of Ref. 34 is a Fabry-Perot. Both studies, however, deal with the competition between longitudinal and transverse modes. In our case the instability is accompanied by temporal oscillations with a frequency of the order of the radial intermode spacing; in the case of Ref. 34, the coexisting transverse and longitudinal modes are synchronous with one another and produce stationary spatial patterns.

### Appendix A. Derivation of the boundary conditions (3.8d)

With reference to Fig. 1, we represent the boundary conditions with the formal statement

$$E(r, -L/2, t) = \mathfrak{R} \wp_{1 \rightarrow 3'} E(r, L/2, t - \frac{\Lambda-L}{c}) \quad (\text{A.1})$$

where  $E$  is the cavity field, the coordinates  $z=-L/2$  and  $z=L/2$  correspond to positions 3' and 1, respectively,  $\wp_{1 \rightarrow 3'}$  is a propagation operator between the indicated points and  $\mathfrak{R}$  is a reflection operator that accounts both for the reduction in the field amplitude and possible curvature effects upon reflection.

Next we introduce the slowly varying field amplitude as in Eq. (3.3) and, with the help of Eq. (A.1), we obtain

$$F(r, -L/2, t) = e^{i\omega_0 \Lambda/c} \mathfrak{R} \wp_{1 \rightarrow 3'} F(r, L/2, t - \frac{\Lambda-L}{c}) \quad (\text{A.2})$$

Because we are interested in deriving boundary conditions for the modal amplitudes  $f_p(z, t)$ , we let

$$F(r, z, t) = e^{-i\delta\omega t} \sum_p A_p(r, z) f_p(z, t) \quad (\text{A.3})$$

and, from Eq. (A.2), we obtain

$$\sum_p A_p(r, -L/2) f_p(-L/2, t) = e^{i\omega_0 \Lambda/c} e^{i\delta\omega \frac{\Lambda-L}{c}} \mathfrak{R} \wp_{1 \rightarrow 3'} \sum_p A_p(r, L/2) f_p(L/2, t - \frac{\Lambda-L}{c}) \quad (\text{A.4})$$

The action of the reflection operator is to scale each modal amplitude  $f_p$  by an amount  $\sqrt{R}$ , where  $R$  is the power reflection coefficient, and to apply a phase curvature to the modal functions  $A_p$ . If we assume, for simplicity, that the phase mirrors are ideal reflectors and that the curved mirrors have the same reflection coefficient, Eq. (A.4) can be written in the form

$$\sum_p A_p(3') f_p(-L/2, t) = e^{i\omega_0 \Lambda/c} e^{i\delta\Omega \frac{\Lambda-L}{c}} R \sum_p f_p(L/2, t - \frac{\Lambda-L}{c}) \quad \Re \varphi_{1 \rightarrow 3'} A_p(1) \quad (A.5)$$

where  $A_p(1)$  and  $A_p(3')$  denote the modal functions at the indicated points. On the other hand, from the properties of the modal functions (See Section 2) we have

$$\Re \varphi_{1 \rightarrow 3'} A_p(1) = A_p(3') e^{-i\omega_p \Lambda/c} \quad (A.6)$$

where  $\omega_p$  is defined by Eq. (2.27), so that Eq. (A.5) can also be expressed in the form

$$\sum_p A_p(3') f_p(-L/2, t) = e^{i\omega_0 \Lambda/c} e^{i\delta\Omega \frac{\Lambda-L}{c}} R \sum_p f_p(L/2, t - \frac{\Lambda-L}{c}) \times A_p(3') e^{-i\omega_p \Lambda/c} \quad (A.7)$$

This leads to the required result

$$f_p(-L/2, t) = R f_p(L/2, t - \frac{\Lambda-L}{c}) e^{-i\delta_p} \quad (A.8)$$

where we have used the definition  $\delta_p = (\omega_p - \omega_0) \Lambda/c$ .

## Appendix B. Derivation of Eqs. (3.18)

The starting point is provided again by Eqs. (3.10) and (3.11) for the modal amplitudes  $f_p$  and by Eq. (3.8d) for their boundary conditions. It is convenient to introduce a new set of stationary amplitude functions  $g_p$  such that

$$f_p(\eta) = g_p(\eta) \exp(i \frac{\bar{\delta}\Omega}{v} \eta) \quad (B.1)$$

Their space-dependent equation is

$$\frac{\partial g_p}{\partial \eta} = \alpha L (1 - i\bar{\Delta}) \int_0^\infty d\rho \rho A_p^*(\rho, \eta) \chi(\rho, \eta) \frac{\sum_{p'} A_{p'} g_{p'}}{1 + \bar{\Delta}^2 + |\sum_{p'} A_{p'} g_{p'}|^2} \quad (B.2)$$

and the boundary conditions are

$$g_p(-1/2) = R g_p(1/2) \exp(i [\bar{\delta}\Omega \gamma_\perp \Lambda/c - \delta_p]) \quad (B.3)$$

Now we assume that the quantity

$$\varphi_p = \bar{\delta}\Omega \gamma_\perp \frac{\Lambda}{c} - \delta_p, \quad p = 0, 1, 2, \dots \quad (B.4)$$

is of the order of  $T$  for some integer value  $p = \bar{p}$ . In this case the boundary conditions (B.3) can be cast into the approximate form

$$(p = \bar{p}); \quad g_{\bar{p}}(1/2) - g_{\bar{p}}(-1/2) = T \left(1 - i \frac{\varphi_{\bar{p}}}{T}\right) g_{\bar{p}}(1/2) \quad (B.5a)$$

$$(p \neq \bar{p}); \quad g_p(1/2) - g_p(-1/2) = (1 - R e^{i\varphi_p}) g_p(1/2) \quad (B.5b)$$

At the same time the spatial integral of Eq. (B.2) is

$$g_p(1/2) - g_p(-1/2) = \alpha L (1 - i\bar{\Delta}) \int_{-1/2}^{+1/2} d\eta \int_0^\infty d\rho \rho A_p^* \chi \frac{\sum_{p'} A_{p'} g_{p'}}{1 + \bar{\Delta}^2 + |\sum_{p'} A_{p'} g_{p'}|^2} \quad (B.6)$$

If we combine Eqs. (B.5) and (B.6) we obtain

$$(p = \bar{p}); \quad g_p(1/2) - g_p(-1/2) = T \left(1 - i \frac{\phi_{\bar{p}}}{T}\right) g_p(1/2) = \alpha L (1 - i \bar{\Delta})$$

$$\int_{-1/2}^{+1/2} d\eta \int_0^\infty d\rho \rho A_{\bar{p}}^* \chi \frac{\sum_{p'} A_{p'} g_{p'}}{1 + \bar{\Delta}^2 + \left| \sum_{p'} A_{p'} g_{p'} \right|^2} \quad (B.7a)$$

$$(p \neq \bar{p}); \quad g_p(1/2) - g_p(-1/2) = (1 - R e^{i\phi_p}) g_p(1/2) = \alpha L (1 - i \bar{\Delta})$$

$$\int_{-1/2}^{+1/2} d\eta \int_0^\infty d\rho \rho A_p^* \chi \frac{\sum_{p'} A_{p'} g_{p'}}{1 + \bar{\Delta}^2 + \left| \sum_{p'} A_{p'} g_{p'} \right|^2} \quad (B.7b)$$

Equations (B.7) lead to the following conclusions:

i) The difference

$$g_p(1/2) - g_p(-1/2)$$

is of the order of  $T$ , so that  $g_{\bar{p}}$  is uniform along the axial direction;

ii) The amplitudes  $g_p$  with  $p \neq \bar{p}$  are of the order of  $T$ .

Hence, in the uniform field limit a possible steady state solution is characterized by the modal amplitudes  $g_{\bar{p}} = O(1)$  and  $g_{p \neq \bar{p}} = O(T)$  and the steady state is governed by the equation (3.18) which generalizes the case of the  $TEM_{00}$  steady state described by Eq. (3.17).

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**Table 1**

Values of the parameters used in the figures

$f = L/\Lambda$	= 0.2	all figures
$L_A/L$	= 0.2	all figures
R (reflectivity)	= 0.9	all figures
$\rho_0$ (scaled radius of curvature of the mirrors)	= 5.0 = 4.1 = 8.0	Figs. 2,3,4,5,7,9,10,11 Fig. 6 Figs. 12,13
$\alpha_1$ (scaled free spectral range)	= 1.0 = 2.0 = 0.5 variable	Figs. 2,3,4,5,6,12,13 Fig. 7 Fig. 11 Figs. 9,10
$\delta_{AC}$ (scaled detuning)	= 0.0 = 0.3 variable	Figs. 2,3,5,6,9,10,11 Fig. 4 Figs. 7,12,13
$\sqrt{\eta_0}$ (beam waist)	= 2.14 = 3.36 = 1.99	Figs. 2,3,4,5,7,9,10,11 Fig. 6 Figs. 12,13
$\delta\omega_1$ (scaled radial frequency spacing)	= 0.74 = 0.91 = 1.48 = 0.37 = 0.54 variable	Figs. 2,3,4,5 Fig. 6 Fig. 7 Fig. 11 Fig. 12,13 Figs. 9,10
$\psi = 2\rho_P/\sqrt{\eta_0}$	= 10.0 = 1.7 variable	Figs. 3,4,6,7,9,10,11,12,13 Fig. 5 Fig. 2

Table 2

<u>Symbol</u>	<u>Meaning</u>
$2C$	Unsaturated gain parameter
$\bar{\delta}_{AC}$	Difference between the atomic transition frequency and the reference cavity resonance in units of $\gamma_{\perp}$ .
$\eta_0$	Rayleigh range of the field profile between the curved mirrors.
$\psi=2\rho_p/\sqrt{\eta_0}$	Ratio between the waist of the pump medium and the beam waist in the region between the curved mirrors.
$\bar{\delta}\omega_1$	Separation between the (n,1) and the (n,0) cavity resonances in units of $\gamma_{\perp}$ .
$\bar{\alpha}_1$	Cavity free spectral range.
$R$	Power reflectivity coefficient of the curved mirrors.
$\bar{\gamma}$	Ratio between the population and polarization relaxation rates.
$L_A/L$	Ratio between the length of the active medium and the distance between the curved mirrors.

## Figure Captions

1. Schematic representation of the ring cavity. The entire length of the folded resonator is  $\Lambda$ ; the length of the active medium is  $L_A$  and the separation between the spherical mirrors is  $L$ . Point 0 marks the origin of the reference system that measures the position along the axis of the cavity, while point 4 denotes the end of a full loop. Points 1 and 1', 3 and 3' denote the longitudinal positions just before and after reflection, respectively. Point 2 is located at half a round trip.
2. Ordinary laser threshold gain plotted as a function of the parameter  $\psi = 2\rho_P/\sqrt{\eta_0}$  for (a) a gain profile of the type given by Eq. (3.5b) and (b) for the gain profile of Eq. (3.5c). As expected, an increase in the transverse dimension of the excited volume leads to a lower threshold gain in both cases.
3. Dependence of the steady state output intensity on the gain parameter  $2C$  [Eq. (3.18a)] for (a) mode  $p=0$ , (b) mode  $p=1$ , (c) mode  $p=2$ . The gain profile of the medium corresponds to Eq. (3.5b). The larger output intensity acquired by mode  $p=1$  relative to  $p=0$  for  $2C$  greater than about 2 is a consequence of the larger modal volume of the higher order modes which provides a better match with the pumped medium.
4. Same as Fig. 3 but with a detuning parameter  $\delta_{AC} = 0.3$  which is 30% of the chosen free spectral range ( $\tilde{\alpha}_1=1$ ) and almost one half of the separation between adjacent radial modes ( $\delta\omega_1=0.74$ ).
5. Dependence of the steady state output intensity on the gain parameter  $2C$  [Eq. (3.18a)] for a medium characterized by the gain profile (3.5b) for curves (a) and (b), and by the profile (3.5c) for curve (c). Curves (a) and (b) correspond to  $p=0$  and  $p=1$ , respectively; curve (c) corresponds to  $p=0$ . The chosen value of  $\psi$  ( $\psi=1.7$ ) simulates a thin column of active medium. As expected, in

this case the presence of an absorbing region at the radial edges causes a significant decrease in the output intensity of the  $p=0$  mode (the curve for the mode  $p=1$  is out of scale).

6. The dependence of the output intensity on the gain approaches the qualitative behavior of the plane-wave theory when the transverse profile of the beam becomes more homogeneous in the radial direction. Here we have increased both the beam waist and the width of the pumped region relative, for example, to the values used in Fig. 4. Note, however, that the parameter  $\psi$  is the same as in Fig. 4 ( $\psi=10$ ). The gain profile is given by Eq. (3.5b).

7. Dependence of the modal intensity on the detuning parameter. Curve (a) shows the behavior of the mode  $(n,p=0)$ ; curve (b) corresponds to the mode  $(n,p=1)$  and curve (c) corresponds to  $(n+1,p=0)$ . Note that the free spectral range is  $\bar{\alpha}_1=2$  and the separation between adjacent radial modes is  $\bar{\delta}\omega_1=1.48$ . The gain profile is given by Eq. (3.5b).

8. The output intensity of a conventional  $\text{CO}_2$  laser as a function of the cavity length. Curves (a) correspond to the laser operating in two consecutive  $\text{TEM}_{00}$  modes and curve (b) to a  $\text{TEM}_{01+10}$  mode. Additional studies will be needed to determine if the observed bistability is the result of hysteresis in the piezoelectric ceramic driver of the cavity mirror, or if it is intrinsic of the dynamics of the system.

9. The dependence of the second laser threshold on the separation between adjacent radial modes for (a)  $\bar{\gamma}=2$ , (b)  $\bar{\gamma}=1$ , and (c)  $\bar{\gamma}=0.1$ . The gain required to produce an instability of the  $p=1$  mode in the presence of a  $p=0$  steady state is measured in units of the ordinary threshold gain. Note that in all cases shown here the instability threshold is only slightly higher than the first laser threshold ( $2C^{(2)}/2C^{(1)}$  is close to unity) when the separation between radial modes becomes progressively smaller. The gain profile is given by Eq. (3.5b).

10. Dependence of the second laser threshold on the radial mode spacing  $\bar{\delta}\omega_1$  for (a)  $p=1$ , (b)  $p=2$ , and (c)  $p=3$ . As expected, the lowest radial modes are the most sensitive to the appearance of unstable behavior. The gain profile is given by Eq. (3.5b).

11. (a) Dependence of the second laser threshold on the parameter  $\psi=2\rho_p/\sqrt{\eta_0}$  for (a)  $p=1$  and (b)  $p=2$ . The pump profile corresponds to Eq. (3.5b).

(b) Same as Fig. 11a with the pump profile given by Eq. (3.5c). These results show that the instability is favored when the cavity is designed to operate with a large value of the parameter  $\psi$ .

12. Dependence of the second laser threshold on the detuning parameter for a number of radial modes; (a) corresponds to  $(\bar{n}, p=1)$ , (b) to  $(\bar{n}, p=2)$ , (c) to  $(\bar{n}-1, p=1)$ , and (d) to  $(\bar{n}-1, p=2)$ . The pump profile is given by Eq. (3.5b).

13. Instability domain (to the right of the solid line) for the radial mode  $(\bar{n}+1, p=0)$ . As shown analytically in Ref. 24, this radial mode cannot become unstable under resonant conditions. For sufficiently large values of  $\bar{\delta}_{AC}$  and appropriate values of the gain parameter, this radial mode becomes unstable through a mechanism that is reminiscent of a phase instability.

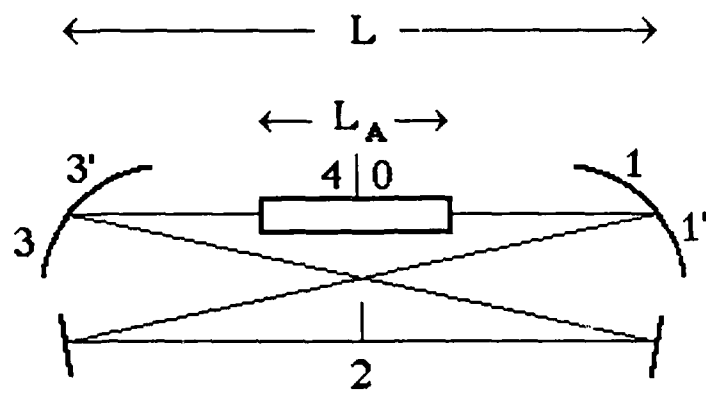


Fig. 1

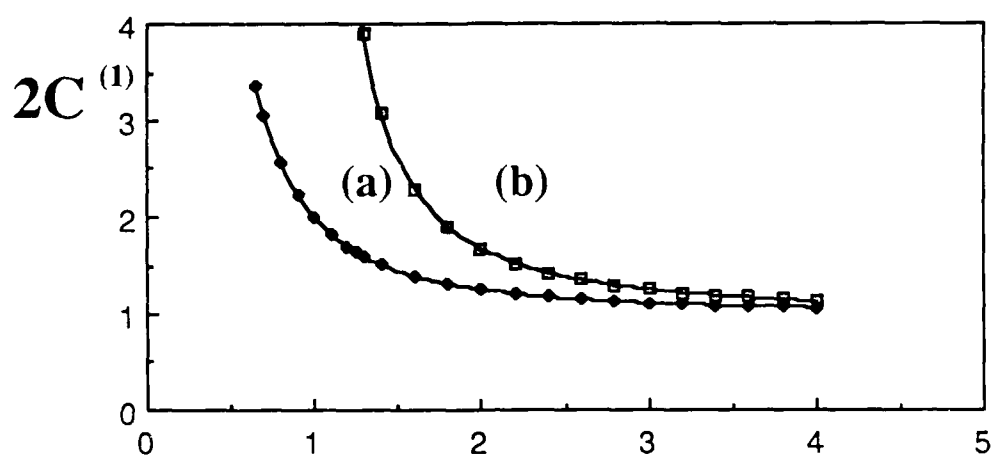


Fig. 2

$\Psi$



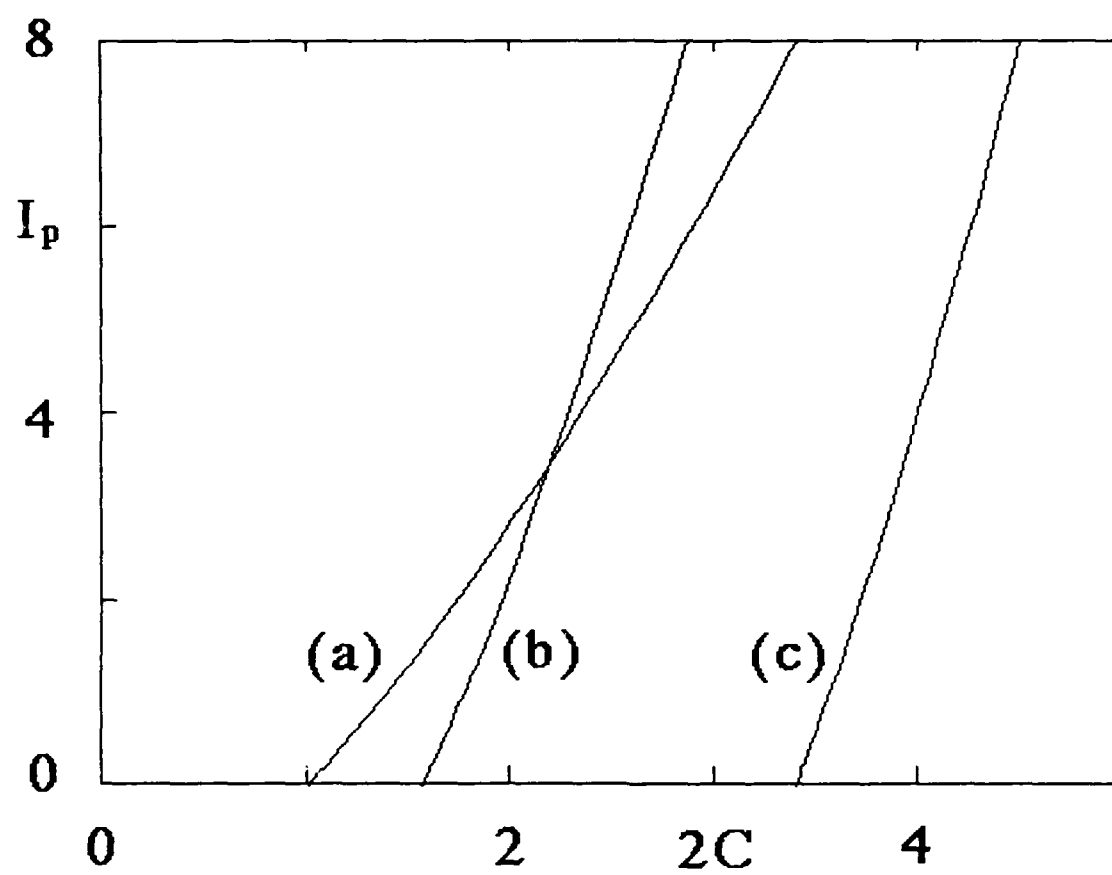


Fig. 3

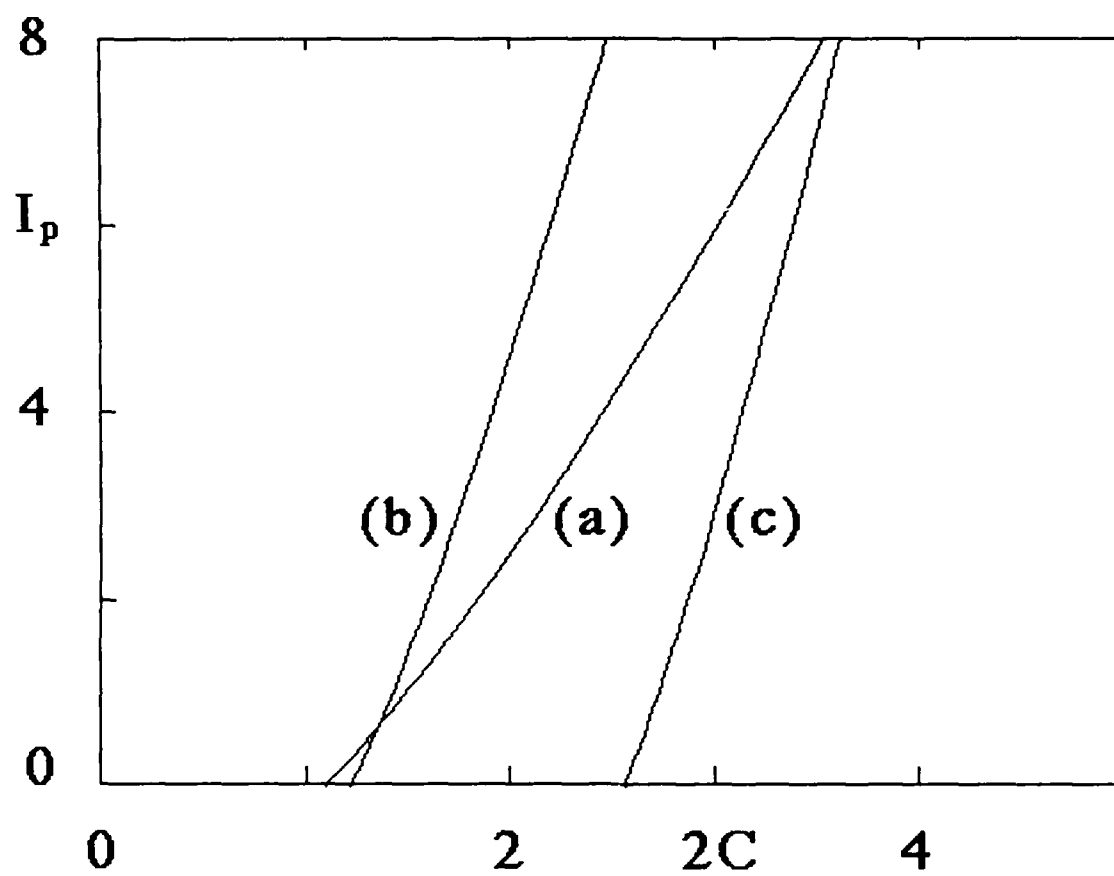


Fig. 4

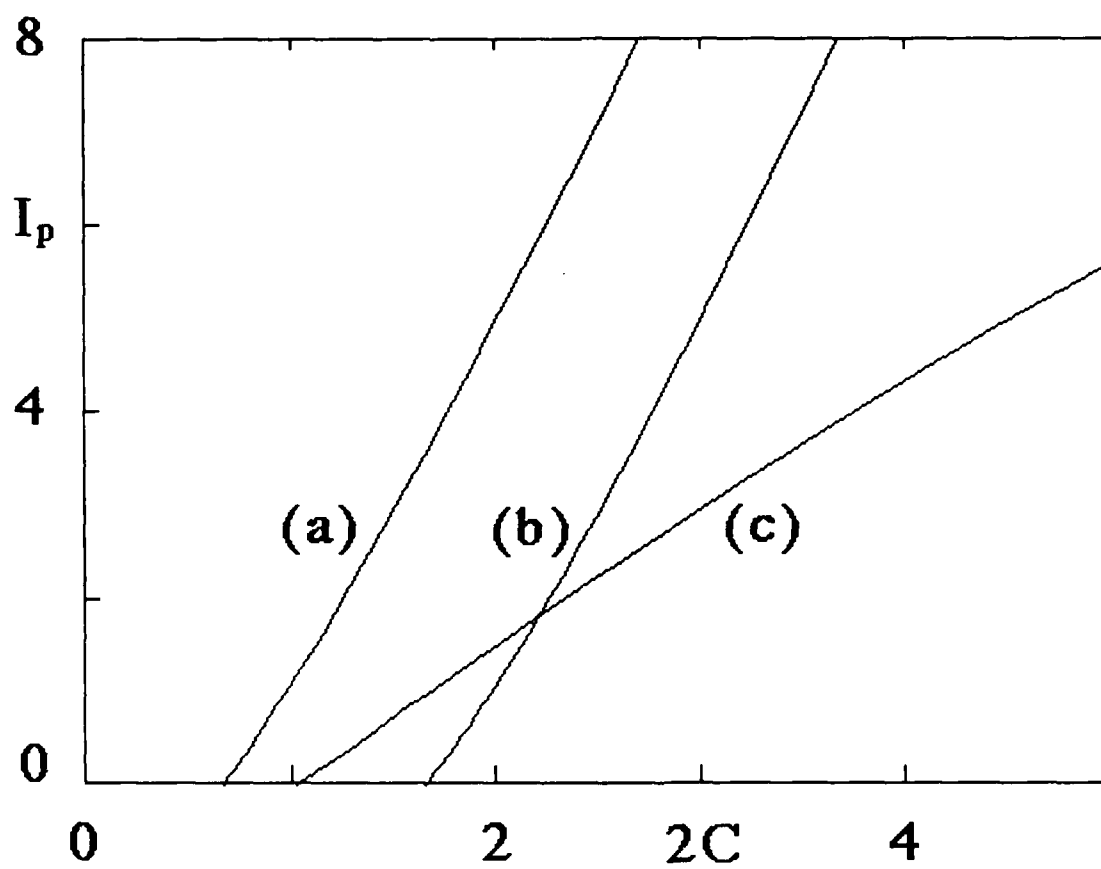


Fig. 5

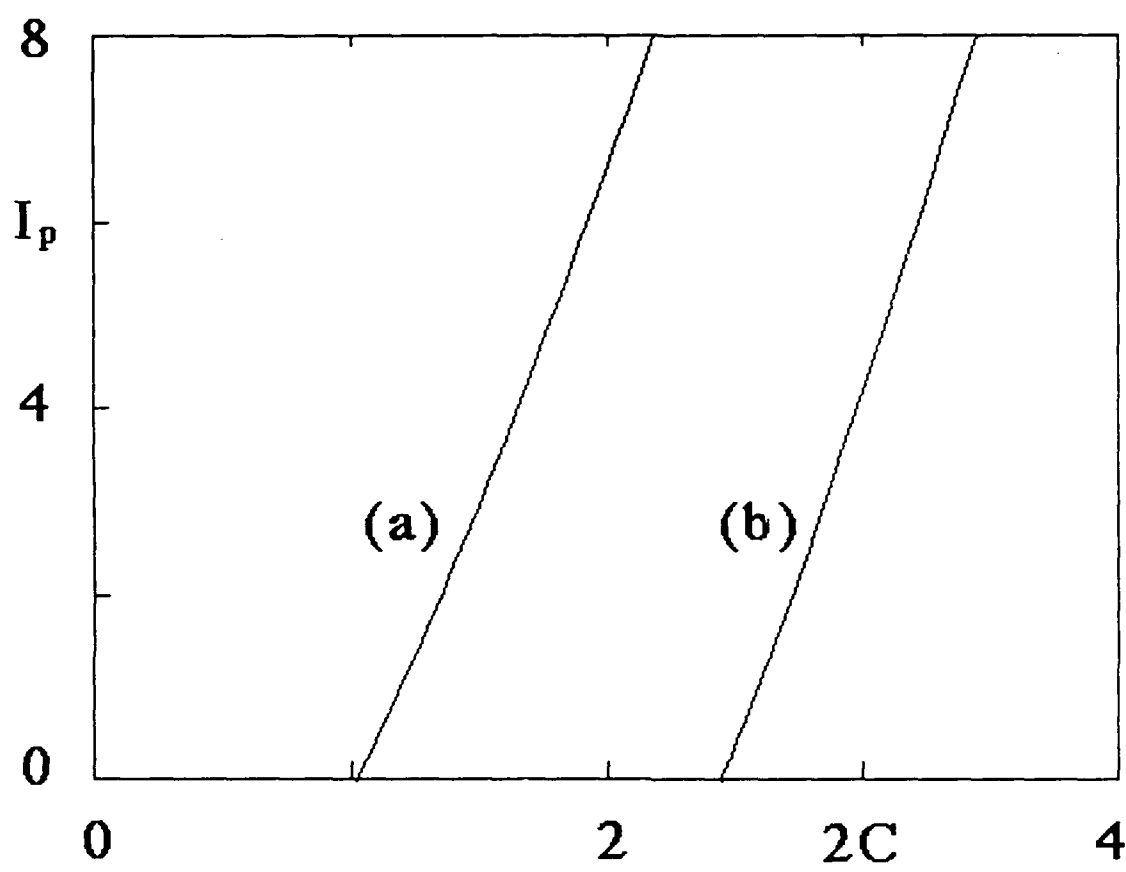


Fig. 6

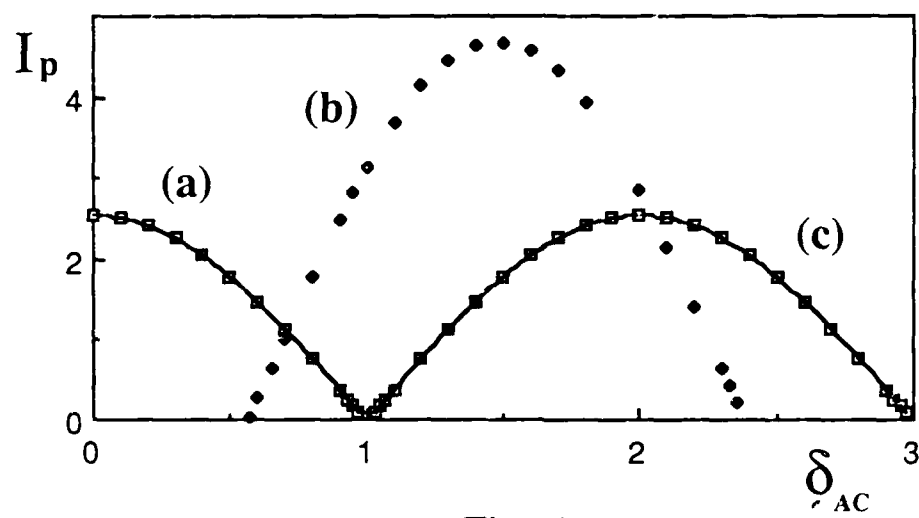


Fig. 7

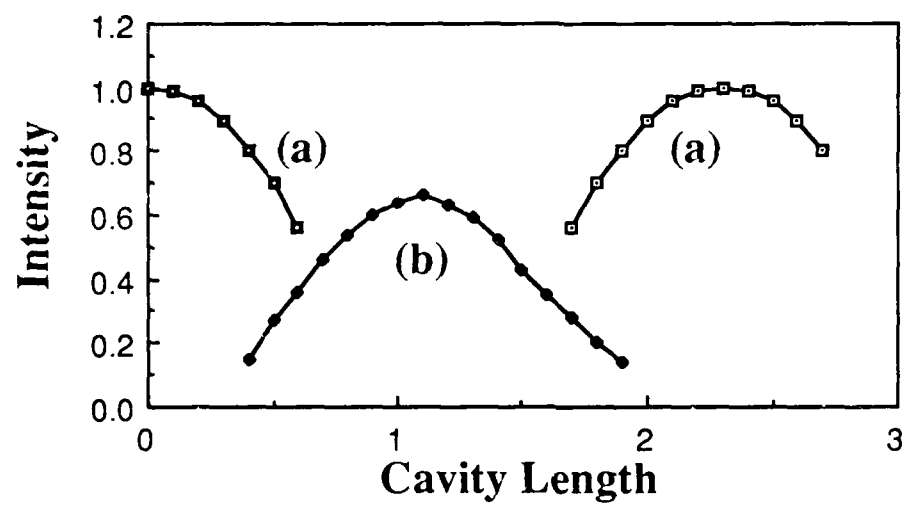


Fig. 8

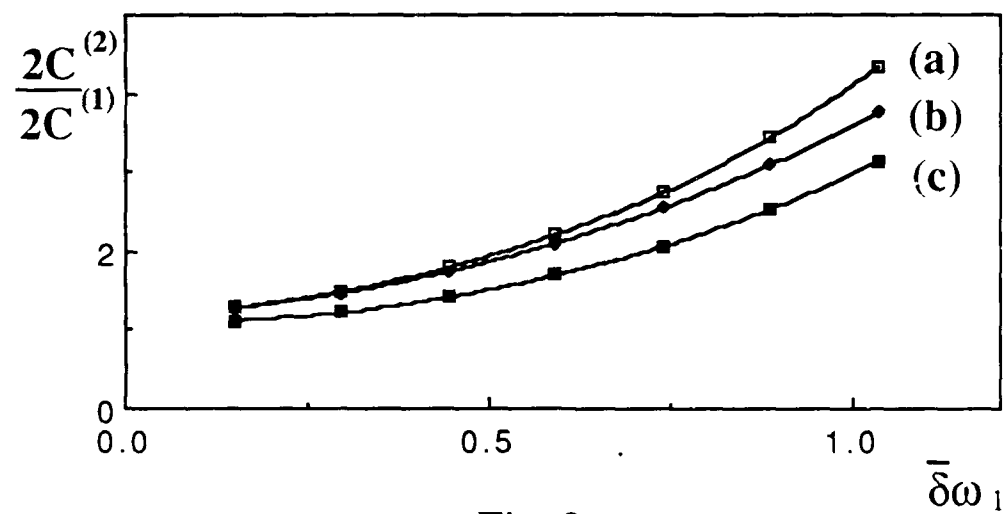


Fig. 9

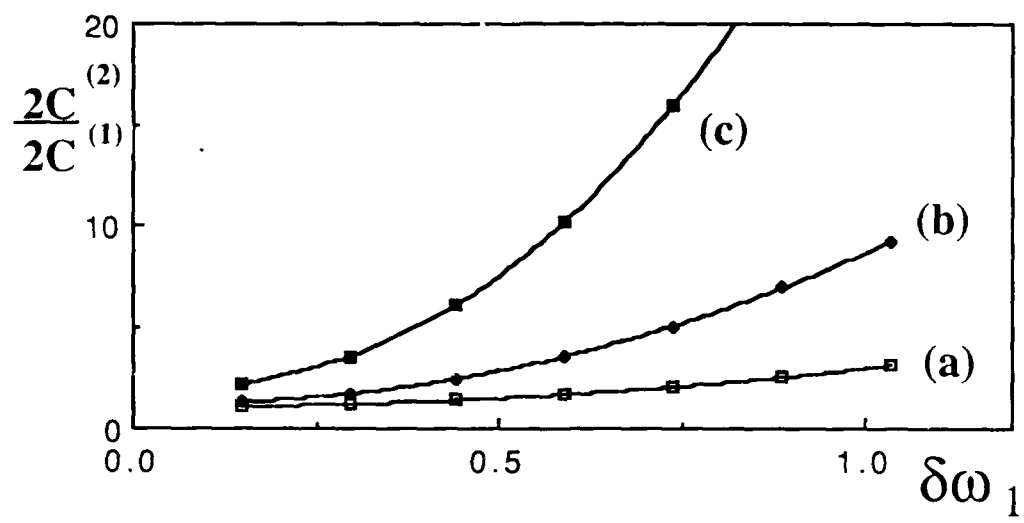


Fig. 10



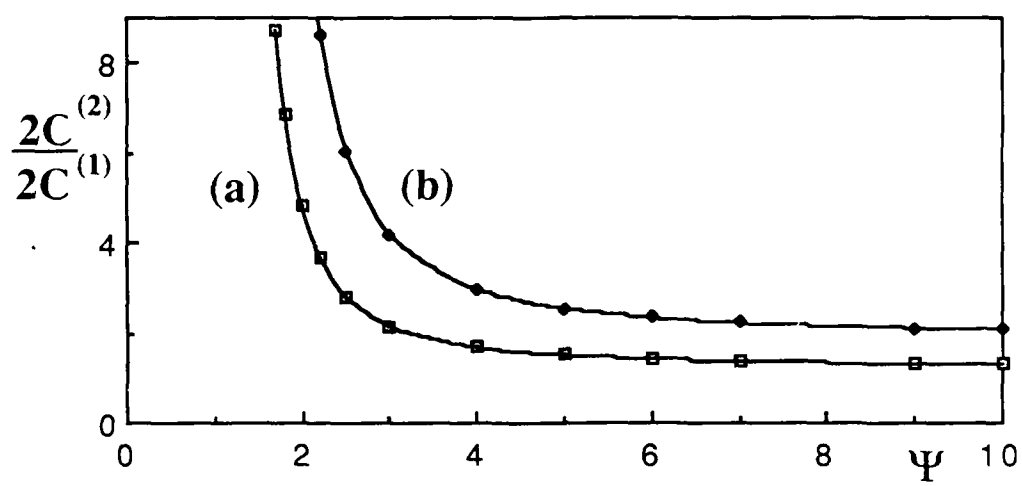


Fig. 11a

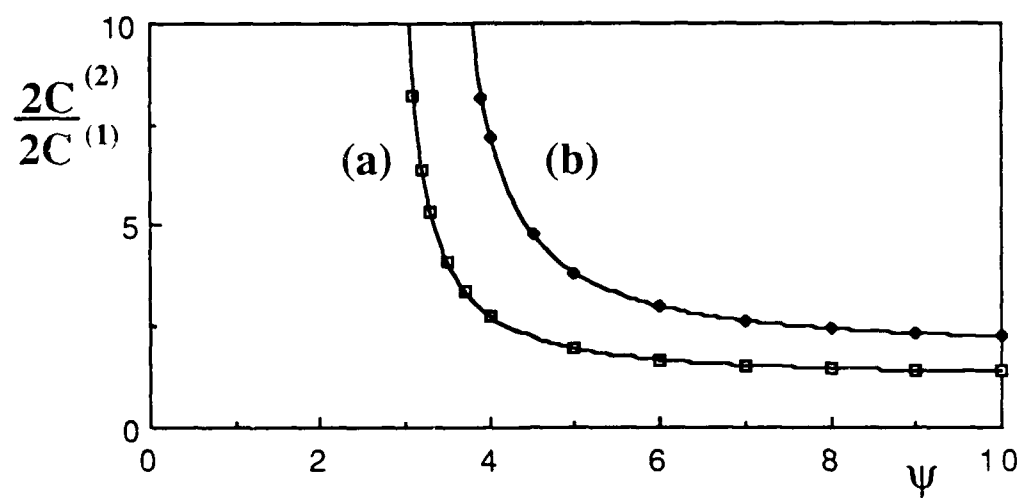


Fig. 11b

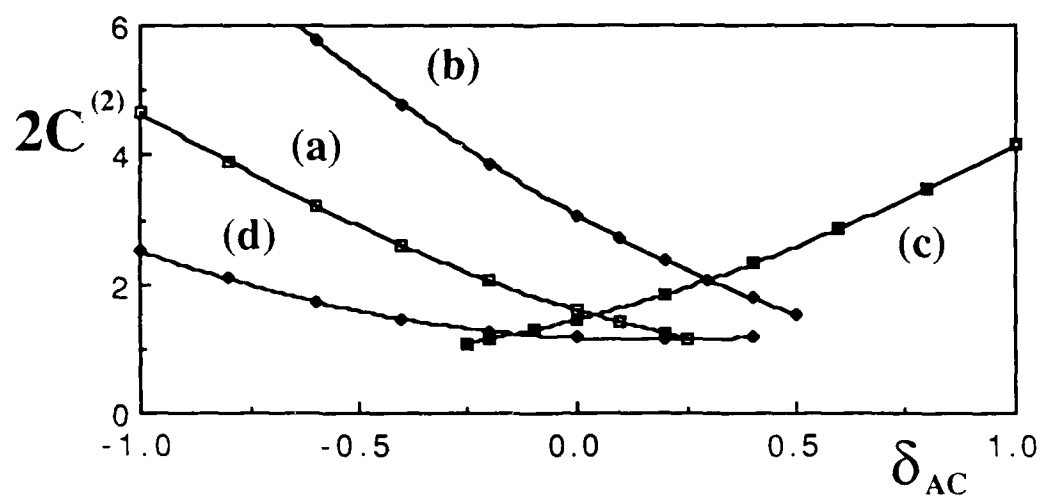


Fig. 12

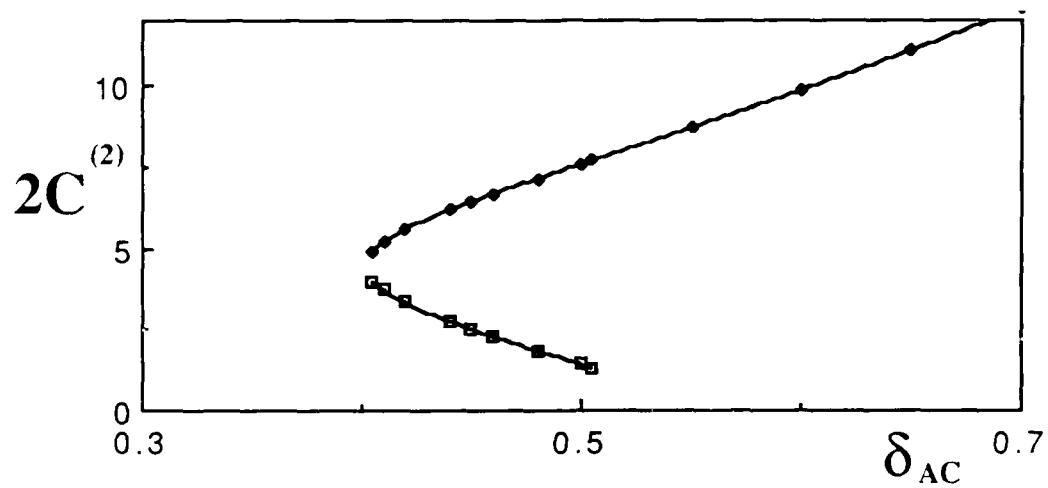


Fig. 13

## Stimulated Raman scattering from hydrogen molecules

We have carried out systematic theoretical studies of stimulated Stokes emissions from gaseous hydrogen molecules irradiated by a frequency-doubled Nd:YAG laser. Our objective is to investigate the evolution of the transverse profiles of the laser and Stokes fields along the sample cell, the spatial distribution of the Stokes emission, and the energy flux exchanges between the two interacting fields. This project was motivated by some experimental results obtained by Dr. Y. S. Liu at the General Electric Company. In his experiments Dr. Liu observed that under tight focusing arrangements a single-mode laser produced a ring pattern of Stokes emission at a pressure of 60 psi. For increasing sample pressure, he also noted the appearance of a central peak in the Stokes radiation, followed by a gradual fading of the ring pattern.

In our theoretical simulation we assume that the sample cell has a cylindrical geometry and that the molecular vibration, approximated by a damped driven classical harmonic oscillator, has reached a quasi-stationary state under the action of the slower varying field envelopes. In this case the steady-state fields equations take the following form:

$$\frac{\partial F_S}{\partial \eta} = \frac{i\alpha}{4} \nabla_{\perp}^2 F_S + \frac{g}{\alpha\beta} \left(1 + \frac{i\delta}{1+\delta^2}\right) |F_L|^2 F_S \quad (1a)$$

$$\frac{\partial F_L}{\partial \eta} = \frac{i}{4\alpha} \nabla_{\perp}^2 F_L - \alpha\beta g \left(1 - \frac{i\delta}{1+\delta^2}\right) |F_S|^2 F_L \quad (1b)$$

In Eqs. (1)  $\alpha$  and  $\beta$  are constants related to the ratios of the frequencies and indices of refraction of the Stokes and laser fields. The symbol  $\delta$  is a detuning parameter the measures the deviation from the resonance transition and  $g$  is a gain factor, proportional to the number density of the molecules.

We have solved this set of equations numerically for various values of the parameters. A brief summary of the results is given below:

1. For a large input beam waist only a central peak is obtained in the Stokes emission at the exit of the sample cell. In this case, generally, the spatial pattern of the transmitted laser field shows both a central peak and a ring-shaped structure. For a smaller input beam waist a ring structure appears in the spatial distribution of the Stokes radiation and the energy transfer is so complete that the laser field is totally depleted.
2. If the Stokes wavelength is in resonance or shifted to the red side of the molecular transition, no ring pattern develops in the Stokes emission. This feature becomes evident only when the Stokes radiation is blue-shifted relative to the molecular transition.
3. In order to simulate the effect of varying pressure we have investigated the dependence of the solutions on the gain parameter. For appropriate selection of the remaining parameters of the system, a ring pattern appears for small gain; for larger gain values the central peak begins to develop and eventually becomes a dominant feature of the spatial distribution of the Stokes emission. These results, as well as item 1, are consistent with Liu's experimental observations.
4. The ring pattern in the Stokes emission may disappear when the input laser intensity becomes sufficiently small. On the other hand this pattern is insensitive to the weak level of the Stokes intensity assumed at the entrance of the sample cell.

## Miscellaneous Theoretical Results

1. We have completed a theoretical and experimental analysis of the behavior of inhomogeneously broadened ring lasers. The results of this investigation include the quantitative analysis of the behavior of the self-pulsing frequency as a function of the most important control parameters and a study of the pulsation waveforms. In a paper published in the Physical Review A by Tarroja, Abraham, Bandy and Narducci we have shown that heterodyne and homodyne spectra together with phase-space plots can be useful in categorizing the periodic and chaotic behavior of such systems. The theoretical model, developed in earlier studies in collaboration with Lugiatto of the University of Milano (presently at the Polytechnic Institute of Torino) is successful in explaining a large number of experimental observations carried out in our laboratories.
2. Albano has implemented new techniques for calculating dimensions and entropies from digitized data. There is increasing evidence that broad-band laser spectra may be a symptom of dynamical chaos, while similarly broad spectra in amplified spontaneous emission may only be the reflection of stochastic noise.
3. Yuan has completed a study of the time evolution equation for the density matrix of a driven Morse oscillator coupled to a bath of harmonic oscillators. The results show that the expected hysteresis, which is typical of classical calculations, does not appear in the context of quantum mechanical calculations. On the other hand, bistability is manifested in the bimodal population distribution attained by the nonlinear oscillator. Distributions of this type have been observed by Bloembergen and others. Time series of the oscillator energy show increasingly more complicated behavior as the driving intensity increases.
4. New results have been obtained by Yuan in the study of the quantum behavior of a dynamical system which is known to exhibit classical chaos. In particular, we have studied the dissociation dynamics of a nondissipative, laser-driven Morse oscillator as a prototype of IR multiphoton dissociation of molecular systems. In the classical treatment of this problem we found primary resonances with winding numbers equal to  $1/n$  (where  $n$  is a small integer), corresponding to the fundamental and overtone transitions of the Morse oscillator. Furthermore, an infinite sequence of secondary resonances are generated as a result of the interaction of 1:1 and 1:2 primary resonances. We have found also that the dissociation half line near the threshold of global instability follows a power law relation, with the critical exponents in close agreement with those predicted by the standard map. This fact, the dissociation rate and the dissociation fractions are amenable to a direct comparison with experimental and quantum theoretical results.
5. We have used calculations of the dimension of strange attractors, correlation functions and higher order coherence properties to analyze and identify the nature of the dynamical or stochastic causes of intensity fluctuations in lasers and amplified spontaneous emission sources. We have obtained experimental data from long amplifiers and unidirectional ring lasers. A preliminary account of these techniques was included in M.H.F. Tarroja's PhD Dissertation.
6. We have developed and analyzed a model for bidirectional homogeneously broadened ring laser such as should describe our previous experiments in  $\text{CO}_2$  lasers. We have found new results for time-dependent periodic and chaotic solutions and we have identified the boundaries for stable operation. We have also extended the traditional models of the single mode homogeneously broadened ring laser to new operating conditions and found new characteristic behaviors. We have analyzed a model of a laser with saturable absorption under conditions in which the absorber and amplifier media have different resonant frequencies and when the cavity is detuned from the center frequency of the active atoms. The results of our calculations predict both bistable and unstable solutions.

7. We have implemented a modification of the Grassberger-Procaccia algorithm for calculating the dimensions of a strange attractor using the singular value decomposition techniques of Broomhead and King. This new method is especially useful in verifying if a dynamic attractor is truly fractal over all scale lengths.

## Experimental Results

1. Following a theoretical suggestion by P. Mandel that a laser swept from below to above threshold would display discontinuous jumps and hysteresis, Tredicce and Scharpf have produced an experimental demonstration of this effect with an  $\text{Ar}^+$  laser. Because the original theoretical proposal was phrased in such terms that a quantitative comparison was not readily possible, Tredicce developed a new theoretical description of this process and obtained good agreement between the experiments and the results of his calculations. These investigations have been published in Optics Communications. Lippi, Hoffer and Abraham, at Bryn Mawr, are investigating the same effects in He-Xe and He-Ne ring and Fabry-Perot lasers.
2. By controlling the level of losses inside the cavity of an  $\text{Ar}^+$  with the help of an acousto-optic modulator, Tredicce has studied the spectral behavior of the laser output. The result of an analysis carried out with a Fabry-Perot interferometer has shown that, as a function of decreasing losses, at first the laser operates in a single mode, then a second mode appears, and finally a third mode also tries to exceed its threshold condition. At this point, usually, chaotic-like behavior sets in. Further investigations are in progress to clarify the nature of the broad-band spectral features.
3. The Drexel Quantum Optics Group has completed construction of two  $\text{CO}_2$  lasers. The first operates with a power output of about 15 Watts, and is used mainly to study transverse effects in laser instabilities. The second laser is a low power (1-2 Watts) sealed-off model which operates as a reference source for heterodyne measurements.
4. We have carried out research to seek improved characterization of both ring and Fabry-Perot laser behaviors. We have measured power spectra of output intensity and digitized the intensity pulsations using the spectrum analyzer and Lecroy fast transient digitizer which have been purchased under the subcontract to Bryn Mawr. Detailed studies of the digitized data have provided new ways to measure chaos using correlation functions and intensity statistics.
5. Gioggia, Abraham and their consultant Lange (from the University of Hanover, FRG) have analyzed the Fabry-Perot laser searching for the recently predicted regions of bistability and instability in the laser frequency near the Lamb-dip region. Considerable progress has been made in understanding the data. A paper containing these results will be published in the special issue of the Optical Society of America devoted to laser instabilities.

## Special Activities

1. Abraham Co-Chaired a Symposium on Chaos which was held in Quebec as part of the 1986 SPIE Meeting. The Symposium ran for two days and featured a program of invited and contributed talks from practically all the major contributors in the area of laser instabilities [Arecchi, Abraham, Boyd, Casperson, Chrostowski (the second Co-Chairman), Lugiato, Harrison, Narducci, Tredicce, Weiss, and several others]. Because many of the participants were also involved in the joint CLEO/IQEC meeting in San Francisco, this Symposium provided the opportunity to fine tune some of the presentations that were scheduled to be given to a much broader audience within a few days.
2. In the Quebec Symposium on Chaos Abraham gave an overview of ways to characterize chaotic phenomena in lasers, and discussed new ways to measure the dimensionality of an attractor and to characterize chaotic behavior in dynamical systems.
3. In the same Symposium Lugiato and Narducci proposed a new classification scheme for labelling the unstable behaviors that arise in laser and laser-related systems based on the concepts of amplitude and phase instability. Their work is limited to the plane-wave approximation, but it represents a significant improvement relative to earlier approaches based on the concepts of single and multimode operation.
4. Tredicce discussed a number of peculiar instabilities observed in CO<sub>2</sub> lasers in collaboration with his Florence colleagues. He attributed the appearance of these pulsations to transverse effects and proposed a phenomenological description based on the concept of intensity-dependent cavity damping rate; the qualitative agreement of the resulting theoretical analysis and the experimental results is quite remarkable. The theoretical analyses leading to these results were carried out jointly by the Bryn Mawr and Drexel groups.
5. Abraham, Mandel, and Narducci have just completed a full length review paper on Laser Instabilities. This article is in press and will appear as part of the series Progress in Optics. The manuscript, 420 typewritten pages long, consists of 10 chapters and more than 800 references.
6. A volume of lecture notes on Laser Physics and Instabilities, by Narducci and Abraham, is in press by the World Scientific Publishers. This work is based on the lectures given by the authors at a Summer School in Laser Physics held in Changchun, China, in August 1985.
7. Abraham, Boyd, and Arimondo organized a Conference entitled Optical Instabilities II which was held in Italy in July 1987. Narducci was a member of the program committee.
8. Narducci was elected Fellow of the American Physical Society for his theoretical contributions to Quantum Optics and especially optical instabilities.
9. J.R. Tredicce and D.K. Bandy are co-editors, with A. Oraevskii of the Lebedev Institute of Moskow, of a special issue of the Journal of the Optical Society of America devoted to recent progress in the nonlinear behavior of laser systems.
10. J.M. Yuan was the organizer of the "Chaos in Molecular Systems" Symposium of the '86 International Laser Science Conference held in Seattle, Washington, Oct. 1986.
11. J.M. Yuan was one of the organizers of the XV International Colloquium on Group Theoretical Methods in Physics held at Drexel University, Oct. 1986.



12. N.B. Abraham, L.M. Narducci and J.M. Yuan gave invited talks at the International Workshop Optical Instabilities II held at Il Ciocco, Italy in July, 1987.
13. N.B. Abraham and L.M. Narducci were among the lecturers at a NATO Summer Institute on Laser Physics.
14. N.B. Abraham was promoted to the rank of Full Profesor at Bryn Mawr College.
15. L.M. Narducci was nominated Francis K. Davis Professor of Physics and Atmospheric Science at Drexel University
16. L.M. Narducci has become a member of the Center for Advanced Studies at the University of New Mexico, Albuquerque.
17. J.M. Yuan was an invited speaker at the 2nd International Laser Science Conference, held in Seattle, Washington, Oct., 1986.
18. J.M. Yuan was an invited speaker at the Telluride Summer Research Institute on Nonlinear Dynamics, June 1987.
19. J.M. Yuan was a Symposium Leader at the Sanibel Symposium held at Marineland, Fla., March, 1987.

## Publications

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"COHERENCE, COOPERATION AND FLUCTUATIONS", edited by F. Haake, L.M. Narducci, and D.F. Walls, Cambridge University Press, 1986.

OPTICAL CHAOS, edited by J.R. Chrostowski and N.B. Abraham, SPIE Proceedings, Vol. 667, SPIE Bellingham, 1986.

### **Presentations**

"Laser Instabilities: An Overview", L.A. Lugiato, and L.M. Narducci, Invited talk at the SPIE Optical Chaos Conference, Quebec, June 1986.

"Bistability and Instabilities in CO<sub>2</sub> Lasers", J.R. Tredicce, L.M. Narducci, F.T. Arecchi, G.L. Lippi, G.P. Puccioni, W. Gadomski, SPIE Optical Chaos Conference, Quebec, June 1986.

"Low-Gain Instabilities in a Model of a CO<sub>2</sub> Ring Laser", L.M. Narducci, J.R. Tredicce, D.K. Bandy, L.A. Lugiato, and N.B. Abraham, Invited talk at the SPIE Optical Chaos Conference, Quebec, June 1986.

"Instabilities in Laser Systems: Plane-wave and Gaussian Transverse Profiles", L.A. Lugiato, L.M. Narducci, J.R. Tredicce, D.K. Bandy, Invited talk at the International Quantum Electronics Conference '86, San Francisco

"Dissipative Quantum and Classical Dynamics: Driven Molecular Vibrations", J.M. Yuan and M. Tung, Second International Laser Science Conference, Seattle, Washington, Oct. 20, 1986.

"Classical, Semiclassical and Quantum Mechanical Methods in Chemical Physics", J.M. Yuan, Physics Department, Swarthmore College, Swarthmore, Pa., Nov. 13, 1986.

"Dissipative Quantum Dynamics: Driven Molecular Vibrations", M. Tung and J.M. Yuan, University of California Summer School in Nonlinear Science, Santa Fe, New Mexico, Aug. 1986.

"Classical Dynamics of Molecular Vibrations", J.F. Heagy, J.M. Yuan, and G.C. Lee, University of California Summer School in Nonlinear Science, Santa Fe, New Mexico, Aug. 1986.

"25 Years of Laser Instabilities", L.M. Narducci, Three invited lectures at the Conference on Quantum Optics and Optical Bistability, Qingdao, People's Republic of China, September 1986.

"Chaotic Dynamical Behavior in Lasers", N.B. Abraham, D.K. Bandy, M.F.H. Tarroja, R.S. Gioggia, S.P. Adams, L.M. Narducci, L.A. Lugiato, VII International Conference on Laser Spectroscopy, Hawaii, June 1986.

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"Dynamic Behavior of a Bistable System with Modulated Input", J.Y. Gao, L.M. Narducci, Annual Meeting of the Optical Society of America, Seattle, October 1986.

"Bidirectional Ring Laser Model: Stability Analysis and Numerical Solutions", N.B. Abraham, G.L. Lippi, L.M. Hoffer, P. Mandel and H. Zeglache, Annual Meeting of the Optical Society of America, Seattle, October 1986.

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"Spontaneous Oscillations in a Single-mode CO<sub>2</sub> Laser in a Fabry-Perot Cavity", G.L. Lippi, N.B. Abraham, J.R. Tredicce, L.M. Narducci, G.P. Puccioni, F.T. Arecchi, Conference on Optical Chaos, Quebec City, Canada, June, 1986.

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"Dynamics of Chaos in Optical Systems", N.B. Abraham, A.M. Albano, M.F.H. Tarroja, Conference on Chaos and Related Nonlinear Phenomena, Jerusalem, 1986.

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"Data Requirements for Reliable Estimation of Correlation Dimension", A.M. Albano, G.C. DeGuzman, A.E. Mees and P.E. Rapp, Wales Conference in Biology, December 1986.

"Correlation Functions and Coherence Properties of Chaotic Laser Emission", 1986 IQEC Meeting, San Francisco.

"Frequency Pushing and Pulling and its Link to Instabilities in Ring and Fabry-Perot Lasers", R.S. Gioggia, N.B. Abraham, M.F.H. Tarroja, D.K. Bandy and L.M. Narducci, 1986 IQEC Meeting, San Francisco.

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Michael B. Rankin, Ph.D., Physics, May 1986

Mingwei Tung, Ph.D., Physics, May, 1986

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